Electromagnetics

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Chapter 1

ELECTROSTATICS

1.1 Introduction

1.1.1 Overview

Many applications of electrical engineering require a knowledge of the behavior of voltages and currents in electronic devices and within conductors. In many other situations it is not enough to understand the behavior the voltages and currents in just the conductors and other components, but also the influence of the voltage and current on surrounding materials. In physics classes we learn that electric and magnetic fields extend beyond the electrical carriers within a device. In electrical and computer engineering the extension of the fields beyond electronic devices and wires can have both beneficial and deleterious effects. Without fields we would not have such modern conveniences as cell
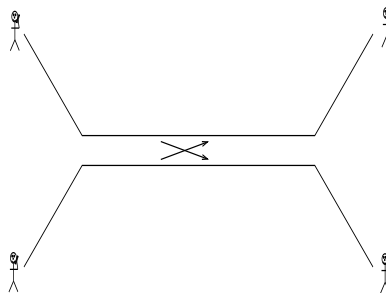

Figure 1.1: Crosstalk between two telephone transmission lines.

phones, television, or even the simplest computer memory chip. On the other hand, unwanted field interactions can cause reversible and irreversible degradation in almost all types of electrical engineering systems. A common example of this type of degradation is evident when a telephone signal on one line leaks over to an adjacent line. This annoying phenomenon is known as *cross talk*. The diagram in Fig. 1.1 shows that the field from one line extends into the other.

In this chapter we examine some of the behavior of electric fields and electric flux. We use the flat panel display as a motivating example for this study. We have chosen the flat panel display as an illustrative example because it shows the ubiquitous nature of electromagnetics in current technology. Flat panel displays are expected to be the video display technology of the future for replacing the current bulky screens on laptop, television, and other applications.

The most common flat panel displays are based on liquid crystal display (LCD) technology. Fig. 1.2 shows a representative cell of a flat panel LCD. Each cell or pixel has a liquid crystal material sandwiched between two transparent conducting plates as shown in Fig. 1.2. The LCD either passes light or blocks light depending on the voltage difference between the two plates. The difference in voltage affects the liquid crystal material by means of the electric field generated between the two plates. Each of the liquid crystal cells represents one of the more than 50,000 individual picture elements or *pixels* on the screen. Each cell is similar to the parallel plate capacitor studied in fundamental physics courses. The parallel plate structure is used throughout this section as a basis for our description of electric fields and electric fluxes.
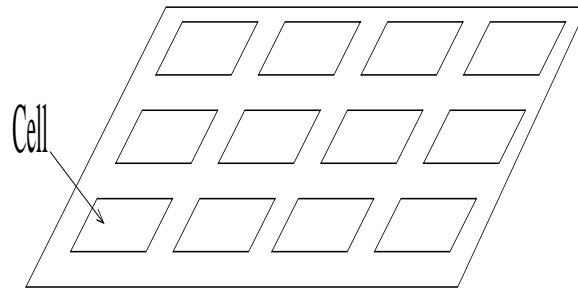
Figure 1.2: A cell in a flat panel LCD display.

Traditionally electrical engineers describe electric fields in terms of vectors. Although vector descriptions of electromagnetic principles are valuable because most engineering students are already familiar with them and because they provide insights into some of the physical properties of fields, vector descriptions of fields are somewhat limited in presenting a complete visual description of fields. This chapter introduces differential forms as a powerful tool for describing and analyzing electric fields. Given that this is likely the reader's first exposure to differential forms, the principles of differential forms are discussed in detail as they are introduced. For both comparison and completeness electromagnetic fields are represented in terms of vectors also.

We introduce differential forms because they provide a powerful and concise mathematical framework for electromagnetics. Differential forms make a clear distinction between electric flux and electric field. They make it simpler to derive theorems and to make coordinate transformations in electromagnetics. However, probably the most important advantage of differential forms at the undergraduate level is that they offer a unique and clear geometric description of electromagnetics not possible using vectors alone. The visual representations that accompany forms are likely to remain in the minds of students whether or not they go on to specialize in electromagnetics or one of its sub-disciplines. These advantages make the additional effort in learning forms worthwhile. Also, students usually find it fun to learn differential forms because forms are elegant and simple to manipulate.

1.1.2   Parallel conducting plates

In this section we focus our attention on the electric fields associated with parallel conducting plates. In the other chapters we shall see that parallel plate transmission lines are often used to describe a variety of important waveguide types. Figure 1.3 shows a general description of parallel conducting plates, the parallel plate capacitor.
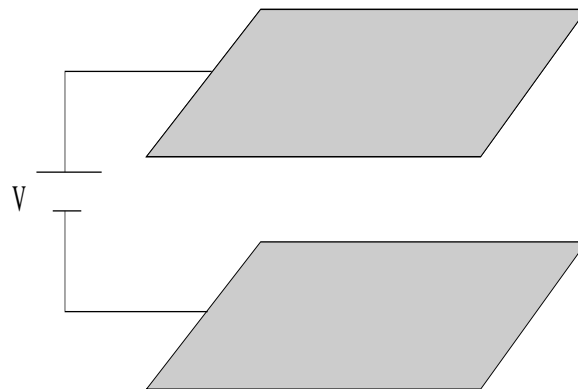

Figure 1.3: A parallel plate capacitor

In this representation we usually consider the separation distance of the plates to be less than one-tenth the length of the conductors. This means that the fields between the plates will not be very different from how they would appear if the plates were infinite in extent. We assume that a potential of 5 volts is applied to the top conducting plate, that the
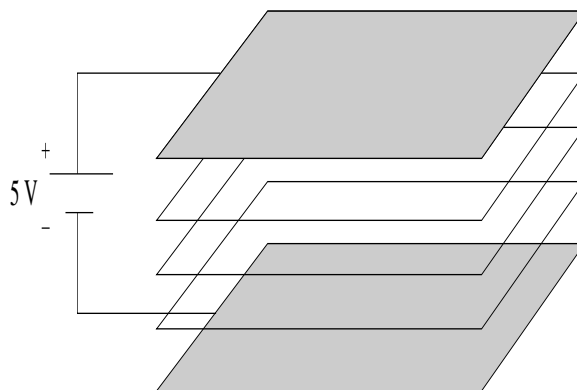
Figure 1.4: The potential between plates of a parallel plate capacitor

lower plate is grounded and that the two plates are separated by 1 mm. For now we assume the material between the two plates is uniform. We notice that the voltage (electric potential) changes with position from the top to the bottom plate. Our first important question is: "What is the potential distribution between the two plates?" It is reasonable to assume that the potential varies from 5 volts at the top plate to 0 volts at the bottom plate in a linear fashion because the material between the plates is the same throughout. We can think of planes of constant voltage between the plates as shown in Fig. 1.4. These planes represent the change in potential from the top plate to the bottom plate. If one follows a path from the top plate to the bottom plate, counting the planes crossed along the way, the number of planes pierced by the path is proportional to the voltage difference between the two conducting plates. The constant of proportionality is the electric field strength in volts per plane. We can express this sum in terms of an integral as

$$V = \int_{\text{bottom}}^{\text{top}} E_x \, dx.$$

The quantity $E_x \, dx$ under the integral sign is a differential form. It is called a 1-form because it has a single variable of integration. The differential form is called the electric field 1-form. In this expression, $E_x \, dx$ is a measure of how much the potential changes per unit distance and has units of V/m. In this case $E_x = 5 \times 10^3$ V/m. The planes in Fig. 1.4 are a convenient geometrical representation of the electric field 1-form. The spacing of the planes indicates the strength of the field; the higher the field the more closely spaced the planes.

In three dimensional space four degrees of forms exist, 0, 1, 2, and 3-forms. Each of these forms has several important examples in electromagnetics. These forms are used and explained in detail as needed in later sections and chapters. For the parallel plate configuration the electric field only has surfaces perpendicular to the x direction. Similarly, the 1-form surfaces could be perpendicular to the y-axis or z-axis and would then be written in terms of dy or dz, respectively. In the general case a 1-form is a linear combination of these differentials, so the surfaces may be skew to the coordinate axes and curved as shown in Fig. 1.5.

In the differential forms model of parallel conducting plates, not only does a voltage exist on the plates, but an electric field, represented by forms, exists between the plates. This is the equipotential representation of the field, or the *energy picture*. Understanding of fields is also enhanced if one looks at the electric field between the plates from the point of view of what happens to small charged body placed in between the two plates.

Consider the potential difference created between the parallel plates as connected to the voltage source. The voltage source draws electrons away from the top conducting plate leaving excess positive charges on its surface. Likewise the bottom conducting plate has negative charges on its surface. A positive test charge placed between the plates is attracted to the negative plate as shown in Fig. 1.6. This force of attraction is proportional to the strength of the electric field between the plates, is in the direction of the electric field, and is known as the Lorentz force. When using the *force picture* of electric fields it is usually most convenient to use vectors in place of forms. The electric field vector is shown in the figure. Its length represents the strength of the electric field and its direction is indicated by the arrow. Using vector notation the Lorentz force law is expressed as

$$\mathbf{F} = q\mathbf{E} \qquad \text{(Lorentz force law, no magnetic fields)}$$

where q is the charge, $\mathbf{F}$ is the force vector, and $\mathbf{E}$ is the electric field vector.
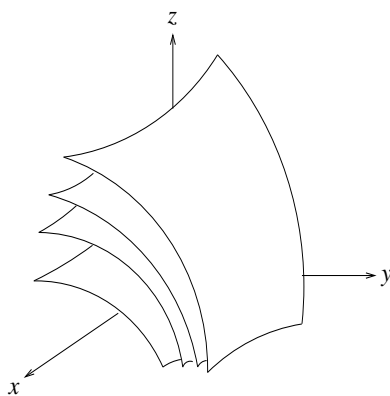
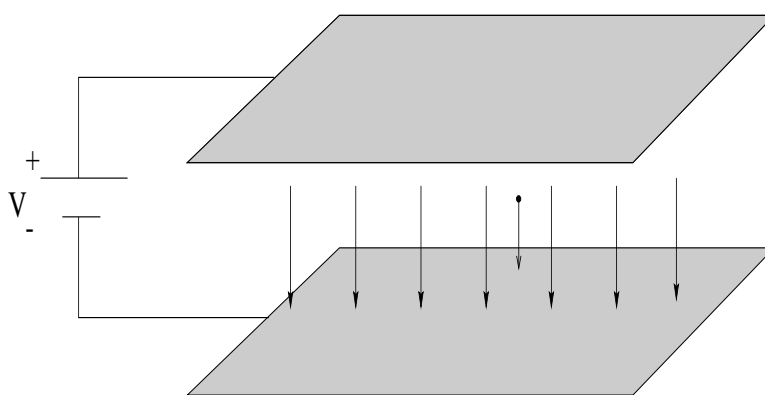**Figure 1.5**: A general 1-form with surfaces that curve in space.

Figure 1.6: A test charge experiences a force due to an electric field.

Electric charge plays an important role in the description of fields using differential forms. From physics we know that with every positive charge there is an associated negative charge. We can view this association as a tube that links or connects a positive charge to a negative charge through intervening material.

For the parallel plate example these tubes are shown connecting positive charges on the top plate to negative charges on the bottom plate. The tubes shown in Fig. 1.7 are the geometrical representation of a 2-form. The 2-forms shown can be expressed as $D_x \, dy \, dz$. This is a 2-form because it has two differential elements. Notice that each tube contains a specified amount of charge. The charge that exists on the plates of the capacitor can be found by counting the flux tubes joining the top and bottom plates. Mathematically this counting is equivalent to integrating the 2-form tubes over the surface area between the plates of the capacitor:

$$Q = \int_{\text{area}} D_x \, dy \, dz.$$

In this representation we see that $D_x \, dy \, dz$ is the charge per tube, so that D_x represents the concentration of charge per unit area.

From the discussion of the graphical representation of 1-forms it is apparent that the 2-form is composed of a 1-form perpendicular to the y-direction and another perpendicular to the z–direction. The connection between charges represented by tubes is called the electric flux density. Flux means flow, and although no physical particles flow from one plate to the other we can think of a stream of influence flowing from one plate to the other as one charge connects through space to its equal and opposite counterpart. The coefficients of a 2-form give the spacing of the tubes, the larger the coefficients are, the more densely packed the tubes become. An arbitrary 2-form has coefficients that are functions of position and the associated tubes may curve and diverge and converge at various points in space.

From the example of the parallel conducting plates it is clear that there is a physical connection between the
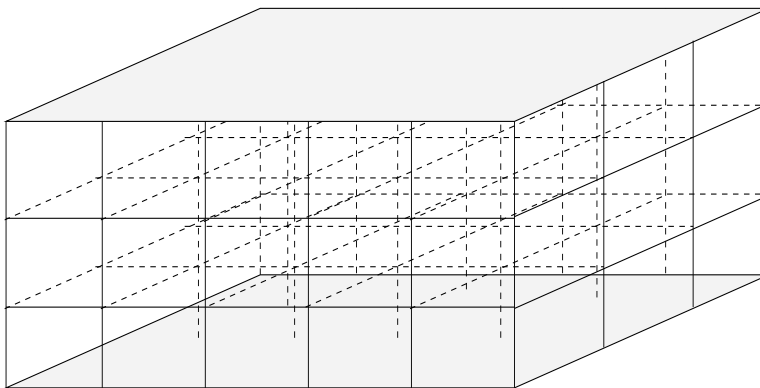
Figure 1.7: Tubes of flux in a parallel plate capacitor

electric field and the electric flux density. We can make geometric and algebraic connections between field and flux using differential forms. Examination of the geometry of electric fields and fluxes shows that the E 1-form planes are composed of the planes that are mutually perpendicular to both of the planes that comprise the D 2-form tubes as shown in Fig. 1.7. In terms of the algebra of forms we require an operator that creates a 1-form from a 2-form and vice versa. This operator is discussed in Sec. 1.5.
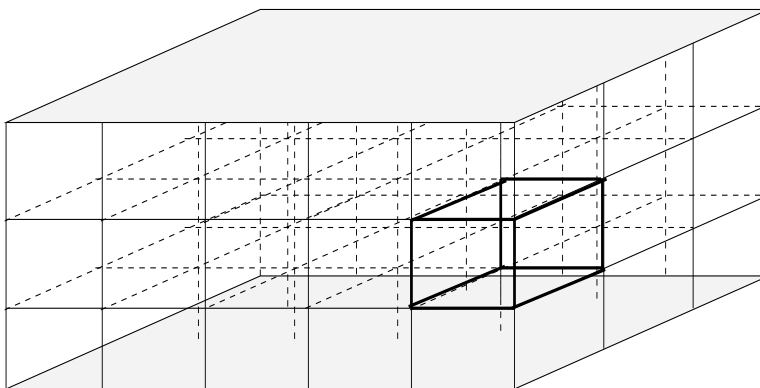

Figure 1.8: Energy density boxes formed by intersection of electric field intensity surfaces and flux density tubes.

To now, we have shown that the electric field may be represent as 1-form planes and that the flux is represented by 2-form tubes. Now let us see what if anything can be made of the boxes formed by combining the field planes and the flux tubes as shown in Fig. 1.8. To find out what those boxes represent we express the combination algebraically as

$$W = \int_{\text{vol}} D_x E_x \, dx \, dy \, dz$$

The combination of the 1-form electric field and the 2-form magnetic flux creates an 3-form entity under the integral sign. Multiplying the dimensional units of D and E gives

$$\frac{\text{C}}{\text{m}^2} \times \frac{\text{V}}{\text{m}} = \frac{\text{J}}{\text{m}^3}.$$

Hence the volume integral of the field multiplied by the flux is the total energy stored in a region of space by the fields present in the region. The 3-form quantity under the integral sign is the energy contained in a cube.

This description of energy density helps us understand why the refresh rate on a flat panel display is limited. Recall that the individual picture elements (pixels) of a flat panel display are illuminated or not depending on the voltage that is applied to them. To switch from one view to another requires that the pixels be changed about 30 times

each second to prevent the eye from seeing a flicker. This means that time is required to move energy to and from the region between the plates to switch from the off state to the on state and any energy stored between the plates must be removed during the process of switching from the on state to the off state. Energy transfer in time is defined as power. Therefore, switching states in a finite amount of time requires power and takes time.

In another view of this we can consider the capacitance of the system and calculate the time required to change states by using the RC time constant of the circuit. It is interesting to note that we can calculate the capacitance from the electric field and the electric flux. The fundamental definition of capacitance is the amount of charge stored given a separation voltage. The capacitance of a single cube defined by the intersection of the tubes of D with the planes of E is simply the quotient, D_x/E_x C/V or Farads.

The examples given in this section have introduced physical descriptions of both electric field and electric flux. Although these examples are simple they present a useful foundation upon which systems with greater mathematical and physical complexity can be built. The following sections of this chapter show how to use these concepts in a more general setting.

1.2   1-forms

The graphical representations described in the introduction are useful in gaining intuitive understanding of the behavior of electromagnetic fields. In order to work analytically with the laws which govern the fields, we must develop a mathematical structure to accompany the graphical representations of the previous section.

As we saw in the previous section, electric field intensity represents potential change with distance. In order to find the total potential difference between two points, we need to integrate the electric field along a path between the points. Graphically, this means that we count electric field intensity surfaces. Mathematically, we must perform a path integral. Quantities which are integrated over paths are called 1-forms.

In the introduction, we discussed the example of a 1-form which represented variation of a field in the x–direction. In general, a 1-form can represent variation in any direction, and can be a combination of differentials of all of the coordinates. An arbitrary 1-form can be written

$$A = A_1 \, dx + A_2 \, dy + A_3 \, dz. \tag{1.1}$$

The three quantities A_1, A_2, and A_3 are the *components* of the 1-form. Two 1-forms A and B can be added, so that

$$A + B = (A_1 + B_1) \, dx + (A_2 + B_2) \, dy + (A_3 + B_3) \, dz. \tag{1.2}$$

1-forms can be integrated over paths. As shown in the introduction, we graphically represent a 1-form as surfaces. The 1-form dx has surfaces perpendicular to the x–axis spaced a unit distance apart. These surfaces are infinite in the y and z directions. The integral of dx over a path from the point $(0,0,0)$ to $(4,0,0)$ is

$$\int_0^4 \, dx = 4.$$

This matches the graphical representation in Fig. 1.9a, since the path shown in the figure crosses four surfaces. If the path were not of integer length, we would have to imagine fractional surfaces in between the unit spaced surfaces. A path from $(0,0,0)$ to $(.25,0,0)$, for example, crosses $.25$ surfaces.

We can also think of dx as a 1-form in the plane. In this case, the picture becomes a series of lines perpendicular to the x–axis spaced a unit distance apart, as shown in Fig. 1.9b. Graphically, integrals in the plane are similar to integrals in three dimensions: the value of a path integral is the number of lines pierced by the path.

In order to graphically integrate a 1-form properly, we also have to think of the surfaces as having an orientation. The integral of the 1-form $- dx$ over a path from $(0,0,0)$ to $(4,0,0)$ is -4. Thus, when we count surfaces pierced by a path, we have to compare the sign of the 1-form with the direction of the path in order to determine whether the surface contributes positively or negatively. The orientation of surfaces can be indicated using an arrowhead on each surface, but since the orientation is usually clear from context, to reduce clutter we do not indicate it in figures.

A more complicated 1-form, such as $3 \, dx + 5 \, dy$, has surfaces that are oblique to the coordinate axes. This 1-form is shown in Fig. 1.10. The greater the magnitude of the components of a 1-form, the closer the surfaces are spaced. For 1-forms with components that are not constant, these surfaces can be curved, as shown in Fig. 1.5. The surfaces can also originate along a line or curve and extend away to infinity, or the surfaces may be finite. In this case, the
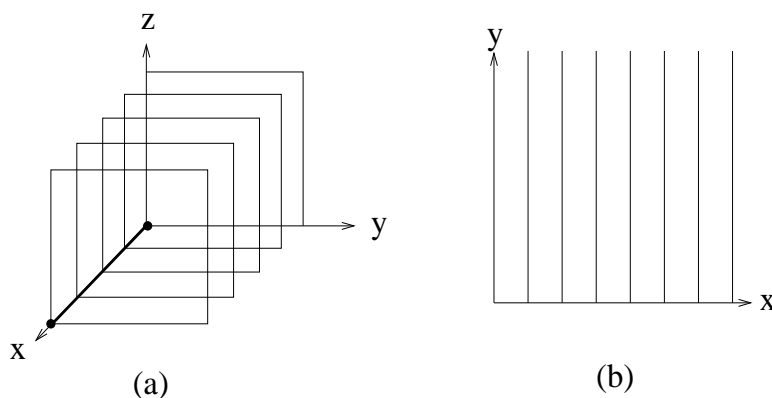
Figure 1.9: (a) The 1-form dx integrated over a path from the point $(0, 0, 0)$ to $(4, 0, 0)$. (b) The 1-form dx in the plane.
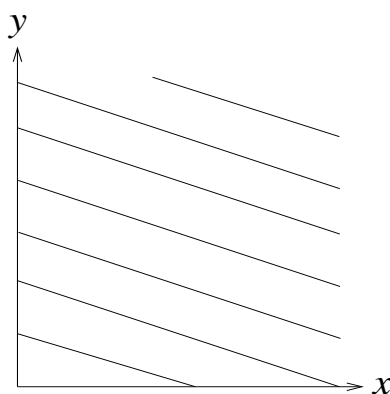

Figure 1.10: The 1-form $3\,dx + 5\,dy$.

integral over a path is still the number of surfaces or fractional surfaces pierced by the path. Some 1-forms are too complicated to be drawn as surfaces in three dimensions (we will give a condition for this later), but any 1-form can be drawn in the plane.

A 1-form represents a quantity which is integrated over a path. A vector represents a quantity with a magnitude and direction, such as displacement or velocity. Despite this difference, both types of quantities have three independent components, and can be used interchangeably in describing electromagnetic field quantities. Mathematically, vectors and differential forms are closely related. In euclidean coordinates, we can make a correspondence between vectors and forms. The 1-form A and the vector $\mathbf{A}$ are equivalent if they have the same components:

$$A_1\,dx + A_2\,dy + A_3\,dz \leftrightarrow A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}} + A_3\hat{\mathbf{z}}. \tag{1.3}$$

We say that the 1-form A and the vector $\mathbf{A}$ are *dual*. Since it is easy to convert between the differential form and vector representations, one can choose the quantity which best suits a particular problem. We will see in the next section that in coordinate systems other then euclidean, the duality relationship between forms and vectors changes.

1.2.1   Curvilinear Coordinates

Many electromagnetics problems have some type of inherent symmetry. In solving problems, it is convenient to choose a coordinate system which reflects that symmetry. For example, the equation which defines a cylinder in rectangular coordinates, $\sqrt{x^2 + y^2} = c$, becomes $\rho = c$ in the cylindrical coordinate system, where ρ is the radial distance from the z–axis.

In general, in three dimensions a coordinate system consists of three functions u, v, and w which assign numbers to each point of space. For convenience, we assume that the directions in which each of the coordinates is changing are perpendicular, so that the coordinate system is orthonormal. In such a coordinate system, the unit differentials are written as $h_1\, du$, $h_2\, dv$, and $h_3\, dw$. The the three functions h_1, h_2, and h_3 are such that the integral over any one of the unit differentials over a path of unit euclidean length in the direction of the particular coordinate is equal to one. For example, if the length of the path from (a, b, c) to (a', b, c) has unit length, then

$$\int_{(a,b,c)}^{(a',b,c)} h_1\, du = 1.$$

These unit differentials correspond to basis vectors according to the relationships

$$
\begin{aligned}
h_1\, du &\quad\leftrightarrow\quad \hat{\mathbf{u}} \\
h_2\, dv &\quad\leftrightarrow\quad \hat{\mathbf{v}} \\
h_3\, dw &\quad\leftrightarrow\quad \hat{\mathbf{w}}.
\end{aligned}
$$

In this section, we give the functions h_1, h_2, and h_3 for two of the most common curvilinear coordinate systems.
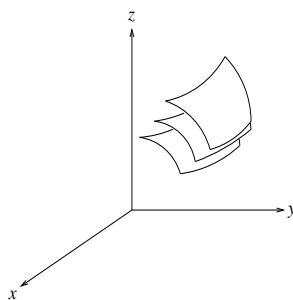

Figure 1.11: The surfaces of unit differentials in general orthonormal curvilinear coordinates are always a unit distance apart.

Cylindrical Coordinates

In the cylindrical coordinate system, a point in space is specified by the radial distance of its x, y coordinates $\rho = \sqrt{x^2 + y^2}$, angle from the $+x$ axis in the x–y plane ϕ, and height in the z direction (Fig. 1.12). Thus, a point is written

$$\boxed{(\rho, \phi, z)} \tag{1.4}$$

in cylindrical coordinates.

The differentials of the cylindrical coordinate system are $d\rho$, $d\phi$ and dz. To convert forms into unit vectors, the angular differential $d\phi$ must be made into a unit differential $\rho\, d\phi$. 1-forms correspond to vectors by the rules

$$
\begin{aligned}
d\rho &\quad\leftrightarrow\quad \hat{\rho} \\
\rho\, d\phi &\quad\leftrightarrow\quad \hat{\phi} \\
dz &\quad\leftrightarrow\quad \hat{\mathbf{z}}
\end{aligned}
$$

Figure 1.13 shows the pictures of the differentials of the cylindrical coordinate system. The 2-forms can be obtained by superimposing these surfaces. Tubes of $dz \wedge d\rho$, for example, are square donut–shaped and point in the ϕ direction.

Spherical Coordinates

In the spherical coordinate system, a point in space is specified by the radial distance from the origin $r = \sqrt{x^2 + y^2 + z^2}$, angle from the $+x$ axis in the x–y plane ϕ, and angle from the z axis θ, as shown in Fig. 1.15. A point is written

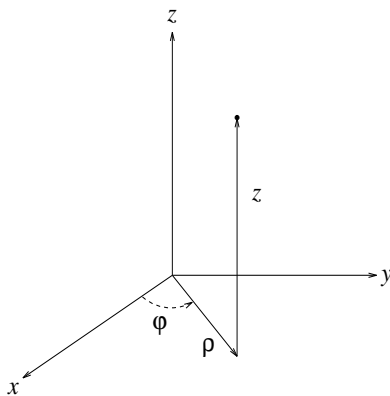$$\boxed{(r, \theta, \phi)} \tag{1.5}$$

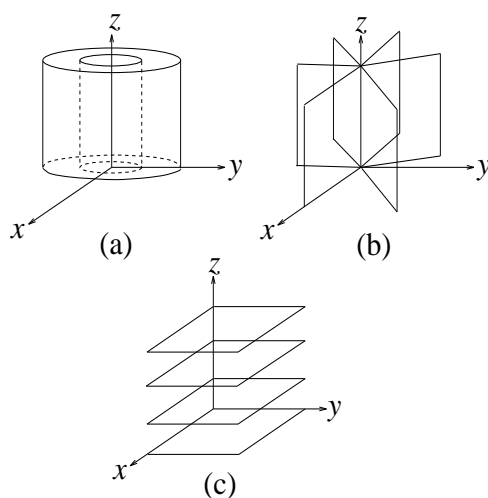**Figure 1.12**: The cylindrical coordinate system.

Figure 1.13: Surfaces of $d\rho$, $d\phi$ scaled by $3/\pi$ and dz.

in these coordinates.

The differentials of the spherical coordinate system are dr, $d\theta$ and $d\phi$. To convert forms into unit vectors, the angular differentials must be made into unit differentials $r\,d\theta$ and $r\sin\theta\,d\phi$. 1-forms correspond to vectors by the rules

$$
\begin{aligned}
dr &\leftrightarrow \hat{\mathbf{r}} \\
r\,d\theta &\leftrightarrow \hat{\theta} \\
r\sin\theta\,d\phi &\leftrightarrow \hat{\phi}
\end{aligned}
$$

Fig. 1.16 shows the pictures of the differentials of the spherical coordinate system.

1.2.2   Integrating 1-forms over paths

The laws of electromagnetics are expressed in terms of integrals of fields represented by differential forms. In order to apply the laws of electromagnetics, we must therefore be able to compute the values of integrals of differential forms. Since 1-forms are by definition mathematical quantities which are integrated over paths, the process of evaluating an integral of a 1-form is very natural. The key idea is that we can replace the coordinates x, y, and z (or u, v, w in curvilinear coordinates), with an equation for a path in terms of a parameter. The parameter of a path is often denoted by t.
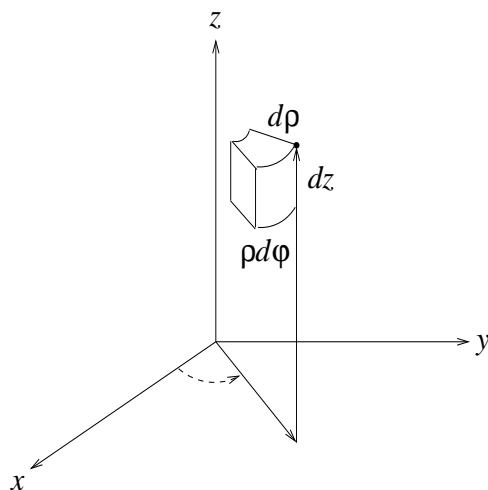
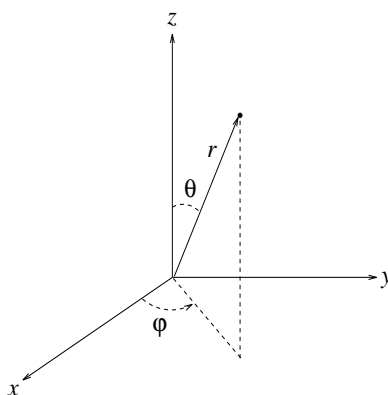**Figure 1.14**: Unit differentials in cylindrical coordinates represented as faces of a differential volume.

Figure 1.15: The spherical coordinate system.

In general, a path is written in the form (f(t), g(t), h(t)), so that the functions f, g, and h give the coordinates of a point on the path for each value of t. We replace the coordinates in a 1-form by these functions, and then the integral can be evaluated. For a differential, when the coordinate is replaced by a function defining the path, we then take the derivative by t to produce a new differential in the variable t. For example, dx becomes $df = f'(t)\,dt$, where the prime denotes the derivative of the function $f(t)$ by t (this operation is a special case of the *exterior derivative*, which will be discussed in a later chapter). The differential form now has a single differential, dt, and the integral can be performed using standard rules of calculus.

Example 1.1. Integrating a 1-form over a path in rectangular coordinates

Consider the 1-form $\alpha = 2\,dx + 3x\,dy$ and a path P which lies along the curve $y = x^2$ from the point $(0,0)$ to $(1,1)$. We wish to find

$$\int_P \alpha. \tag{1.6}$$

We parameterize the path in the variable t, so that the path becomes $(x = t, y = t^2)$, with t ranging from zero to one. We then substitute these values for x and y into the integral,

$$\int_P \alpha(x,y) \quad = \quad \int_0^1 \alpha(t,t^2)$$
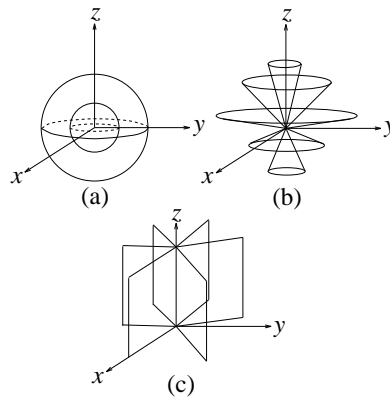
Figure 1.16: Surfaces of dr, $d\theta$ scaled by $10/\pi$ and $d\phi$ scaled by $3/\pi$.
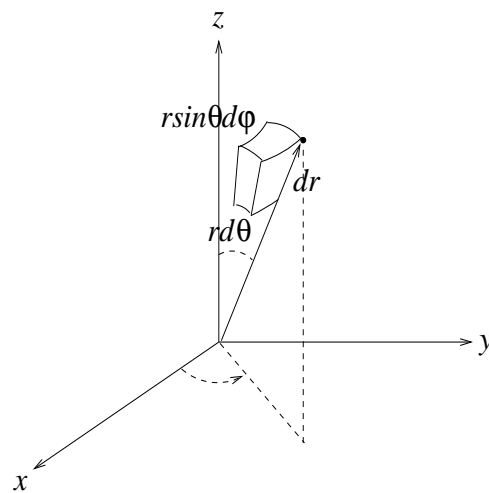
Figure 1.17: Unit differentials in spherical coordinates represented as faces of a differential volume.

$$
\begin{aligned}
&= \int_0^1 \left(2\,dt + 3t\,d(t^2)\right) \\
&= \int_0^1 \left(2 + 6t^2\right) dt \\
&= 4.
\end{aligned}
$$

Example 1.2. Integrating a 1-form over a path in cylindrical coordinates

Suppose we want to integrate the 1-form dx over the unit circle in the x-y plane. We want to change variables from x to ϕ, so that we parameterize the unit circle as $(\cos\phi, \sin\phi, 0)$. The integral is

$$
\begin{aligned}
\int_C dx &= \int_0^{2\pi} d\cos\phi \\
&= -\int_0^{2\pi} \sin\phi\,d\phi
\end{aligned}
$$

$$= \quad 0.$$

We could have guessed the result by noting that each surface of dx pierced by the path in the positive direction (such that the orientation of the surface is the same as the counterclockwise direction of integration along the path) is canceled when the path pierces the same surface in the negative direction.

1.3   2-forms, 3-forms, and the Exterior Product

As we showed in the introduction to this chapter, a 2-form is a quantity which is integrated over a two–dimensional surface. The quantity representing flow of a fluid, for example, has units of flow rate per area, and would be integrated over a surface to find the total flow rate through the surface. Similarly, the integral of electric flux density over a surface is the total flux through the surface, which has units of charge. A general 2-form C is written as

$$C = C_1 \, dy \wedge \, dz + C_2 \, dz \wedge \, dx + C_3 \, dx \wedge \, dy. \tag{1.7}$$

The wedge between differentials is known as the *exterior product*. This product allows one to combine 1-forms to produce differential forms of higher degree. The 2-form $dy \wedge \, dz$, for example, is the exterior product of the 1-forms dy and dz. The exterior product has the important property that if two differentials are interchanged, the sign of the product changes. In other words, the exterior product of 1-forms is antisymmetric. For example, $dy \wedge dz = - \, dz \wedge dy$. Using this property, it is easily seen that the wedge product of two like differentials is zero: $dx \wedge \, dx \, = \, 0$. For convenience, we usually use the antisymmetry of the exterior product to put differentials of 2-forms into right cyclic order, as in Eq. (1.7).

Two 1-forms C and D can be added, so that if C and D are 2-forms, their sum is

$$C + D = (C_1 + D_1) \, dy \wedge \, dz + (C_2 + D_2) \, dz \wedge \, dx + (C_3 + D_3) \, dx \wedge \, dy. \tag{1.8}$$

Like 1-forms, 2-forms have three independent components, and a correspondence between 2-forms and vectors can be made. A 2-form with differentials in right cyclic order can be converted in euclidean coordinates to a vector as follows:

$$C_1 \, dy \wedge \, dz + C_2 \, dz \wedge \, dx + C_3 \, dx \wedge \, dy. \leftrightarrow C_1 \hat{\mathbf{x}} + C_2 \hat{\mathbf{y}} + C_3 \hat{\mathbf{z}}. \tag{1.9}$$

The 2-form C is said to be dual to the vector $\mathbf{C}$.

Example 1.3. Exterior product of 1-forms.

Let $A = 3 \, dx + dy$ and $B = 2 \, dx + 3 \, dy$. Then

$$
\begin{aligned}
A \wedge B \quad &= \quad (3 \, dx + \, dy) \wedge (2 \, dx + 3 \, dy) \\
&= \quad 6 \, dx \wedge \, dx + 9 \, dx \wedge \, dy + 2 \, dy \wedge \, dx + 3 \, dy \wedge \, dy \\
&= \quad 9 \, dx \wedge \, dy - 2 \, dx \wedge \, dy \\
&= \quad 7 \, dx \wedge \, dy.
\end{aligned}
$$

This 2-form is dual to the cross product $(3\hat{\mathbf{x}} + \hat{\mathbf{y}}) \times (2\hat{\mathbf{x}} + 3\hat{\mathbf{y}})$.

Example 1.4. Exterior product of a 1-form and a 2-form.

Let $A = 3 \, dx + \, dy$ and $B = 2 \, dy \, dz + 3 \, dz \, dx$. Then

$$
\begin{aligned}
A \wedge B \quad &= \quad (3 \, dx + \, dy) \wedge (2 \, dy \wedge \, dz + 3 \, dz \wedge \, dx) \\
&= \quad 6 \, dx \wedge \, dy \wedge \, dz + 3 \, dy \wedge \, dz \wedge \, dx \\
&= \quad 9 \, dx \wedge \, dy \wedge \, dz.
\end{aligned}
$$

The result is a 3-form. The coefficient of this 3-form is equal to the dot product $(3\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cdot (2\hat{\mathbf{x}} + 3\hat{\mathbf{y}})$. We will discuss 3-forms in greater detail below.
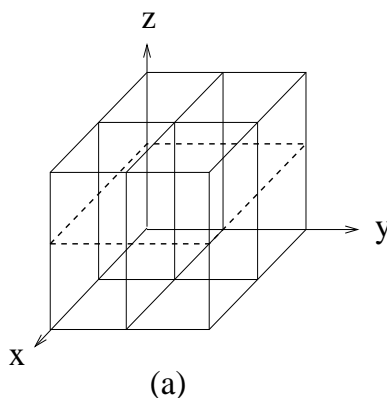
(a)

Figure 1.18: The 2-form $dx \wedge dy$ integrated over a square in the x–y plane of side 2

2-forms are integrated over areas, or two–dimensional regions of space. When a 2-form appears under an integral, we often drop the wedges for conciseness:

$$\int_0^5 \int_0^5 dx \wedge dy \rightarrow \int_0^5 \int_0^5 dx \, dy. \tag{1.10}$$

2-forms are graphically represented as tubes. The picture of $dx \wedge dy$ consists of the surfaces of dx superimposed with the surfaces of dy. The sets of surfaces intersect to form tubes in the z direction. The integral of a 2-form over an area is the number of tubes crossing the area. For $dx \wedge dy$, the integral over a square in the x–y plane of side 2 is 4, and as shown in in Fig. 1.18, nine tubes cross this square. The greater the components of a 2-form are in magnitude, the smaller and more dense are the tubes of the 2-form.

As with 1-forms, the tubes of a 2-form have an orientation. The tubes of $dx \wedge dy$, for example, are oriented in the $+z$ direction, whereas the tubes of $dy \wedge dx$ are oriented in the $-z$ direction. Areas of integration also have an orientation, since their are two possible normal directions for any area. The limits of a double integral specify a direction around the perimeter, and the right–hand rule applied to this direction specifies the orientation of the area. When integrating graphically, we compare the orientation of each tube with the orientation of the area of integration, and the tube counts positively if the orientations are the same, and negatively otherwise.

1.3.1   2-forms in Curvilinear Coordinates

In general curvilinear coordinates, the unit differential for 2-forms are $h_2 h_3 \, dv \, dw$, $h_3 h_1 \, dw \, du$, and $h_1 h_2 \, du \, dv$. If the 2-form $h_2 h_3 \, dv \, dw$ is integrated over a surface which lies in the v-w plane, then the factor $h_2 h_3$ is such that the value of the integral is equal to the area of the surface. The unit differentials are dual to the unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$.

In the cylindrical coordinate system, 2-forms and vectors are related by

$$\begin{aligned} \rho \, d\phi \wedge dz &\leftrightarrow \hat{\rho} \\ dz \wedge d\rho &\leftrightarrow \hat{\phi} \\ d\rho \wedge \rho \, d\phi &\leftrightarrow \hat{\mathbf{z}} \end{aligned}$$

The 2-form $d\rho \, d\phi$, for example, is dual to the vector $(1/\rho)\hat{\mathbf{z}}$.

In the spherical coordinate system, 2-forms and vectors are related by

$$\begin{aligned} r \, d\theta \wedge r \sin\theta \, d\phi &\leftrightarrow \hat{\mathbf{r}} \\ r \sin\theta \, d\phi \wedge dr &\leftrightarrow \hat{\theta} \\ dr \wedge r \, d\theta &\leftrightarrow \hat{\phi} \end{aligned}$$

The 2-form $d\theta \, d\phi$, for example, is dual to the vector $\hat{\mathbf{r}}/(r^2 \sin\theta)$.

Example 1.5. Integrating a 2-form using spherical coordinates

Suppose we want to integrate the 2-form $dy\,dz$ over the hemisphere with $x > 0$ and radius a centered at the origin.

1.3.2   3-forms

With the exterior product, we can combine three 1-forms, or a 1-form and a 2-form, to obtain a 3-form. 3-forms represent densities, such as the density of electric field energy shown in the introduction. Using the antisymmetry of the exterior product, we can always order the product of all three differentials in right cyclic order, $dx \wedge dy \wedge dz$. Any 3-form can be written as

$$q\,dx \wedge dy \wedge dz \tag{1.11}$$

where q is the coefficient of the 3-form. Notice that 3-forms are different from 1-forms and 2-forms, since there is only one component, rather than three.
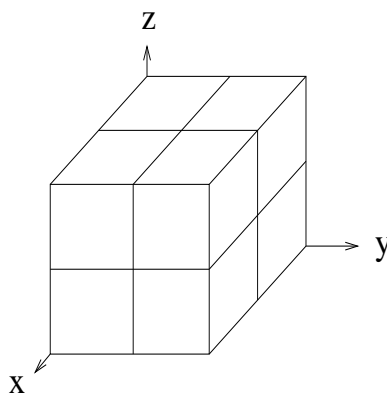

Figure 1.19: The 3-form $dx \wedge dy \wedge dz$ integrated over a cubic region of side two is eight, since there are eight boxes inside the region.

3-forms represent densities, and are integrated over volumes, or three–dimensional regions of space. The graphical representation is boxes. The picture of the 3-form $dx \wedge dy \wedge dz$ consists of the surfaces of the 1-forms dx, dy, and dz superimposed. These sets of surfaces intersect to produce cubes of unit side which fill all space. The integral of this 3-form over any volume is the number of boxes inside the volume. The integral

$$\int_0^2 \int_0^2 \int_0^2 dx\,dy\,dz = 8$$

is graphically represented in Fig. 1.19. (Note that we have dropped the wedges in writing the integral of this 3-form). Each box of a 3-form has a sign associated with it. The integral of $-dx \wedge dy \wedge dz$ over a cube of side two is -8, for example, so that each box of this 3-form contributes minus one to the value of the integral instead of one as was the case for $dx \wedge dy \wedge dz$.

In general, the coefficient of a 3-form is not constant. In this case, the boxes associated with the 3-form can be smaller or larger than those of $dx \wedge dy \wedge dz$. The greater the coefficient, the smaller and more closely packed are the boxes. This reflects the fact that the integral of a 3-form with a greater coefficient should yield a large value.

In curvlinear coordinates, the 3-form $dx\,dy\,dz$ becomes $h_1 h_2 h_3\,du\,dv\,dw$. The integral of this 3-form over any three dimensional region is equal to the volume of the region. In cylindrical coordinates, this 3-form becomes $\rho\,d\rho\,d\phi\,dz$, and in spherical coordinates, $r^2 \sin\theta\,dr\,d\theta\,d\phi$.

1.4 Gauss's Law for the Electric Field

1.4.1 Electric Flux Density

As we saw in the introduction, the electric flux density is represented by a 2-form, which in general has the form

$$D = D_1 \, dy \, dz + D_2 \, dz \, dx + D_3 \, dx \, dy. \tag{1.12}$$

The coefficients of D have units C/m^2, and D has units of D, which is the unit of electric charge. The tubes of D represent flux extending from positive charges to negative charges, or from a charge to infinity. When the electromagnetic field changes in time, as we will see in the following chapter the flux tubes can also form closed loops.

1.4.2 Gauss's Law

Gauss's law for the electric field is written as

$$\oint_S D = \int_V \rho \tag{1.13}$$

where ρ is a 3-form representing the volume density of electric charge (do not confuse the electric charge density with the radial coordinate in the cylindrical coordinate system!). Each box of the 3-form ρ represents 1 Coulomb of charge. S is known as the Gaussian surface. This surface is not physical, but is a mathematical construction used to evaluate Gauss's law. The region V is the interior of the Gaussian surface S.
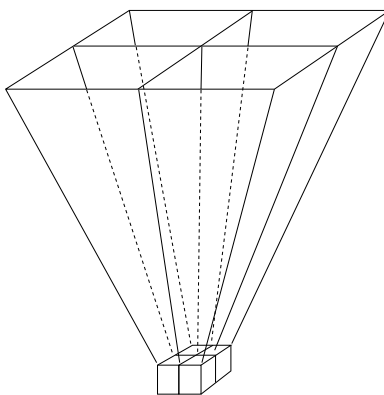

Figure 1.20: A graphical representation of Gauss's law for the electric flux density: cubes of ρ produce tubes of D. In general, the tubes of D will extend out in all directions from the cubes of ρ.

Gauss's law states that for any Gaussian surface, the difference between the number of tubes going into the Gaussian surface and the number of tubes going out must be equal to the number of boxes of charge inside the Gaussian surface. As a consequence, tubes of electric flux begin at positive charges or end at negative charges. As shown in Fig. 1.20, one Coulomb of charge produces precisely one tube of electric flux. The tubes may also extend to infinity or form closed loops, but where they end or begin, charge represented by ρ must be present.

1.4.3 Examples

In this section we will find the electric flux due to a point charge, a line charge and a plane charge using Gauss's law for the electric field. Our goal for each case is to find a 2-form that integrates over any closed surface to yield the amount of charge contained in the surface.

We first use the symmetry of the source to guess the direction of the tubes of flux emanating from the charges. This tells us which differentials appear in the electric flux density 2-form D. Once we know the direction of the tubes, all we need to find is the constant that multiplies the differentials in D.

We then pick a convenient Gaussian surface to use in applying Gauss's law. Since the integral of the flux over the surface must equal the amount of charge inside the surface, we can find the constant in D, and the problem is solved.

Point Charge

The tubes of flux from a point charge Q extend out radially from the charge (Fig. 1.21). Thus, the flux density 2-form D has to be a multiple of the differentials $d\theta\, d\phi$ in the spherical coordinate system. The tubes of $d\theta\, d\phi$ are denser near the poles $\theta = 0$ and $\theta = \pi$ than at the equator. We need to include the correction factors $r^2 \sin\theta$ to have tubes with the same density everywhere in space. Thus, D has the form

$$D_0 r^2 \sin\theta\, d\theta\, d\phi \tag{1.14}$$

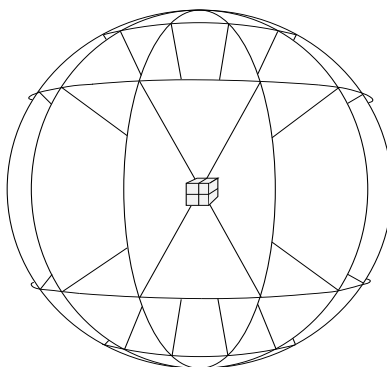where D_0 is some constant we need to find.

Figure 1.21: Electric flux density due to a point charge. Tubes of D extend out radially from the charge.

To apply Gauss law $\oint_S D = \int_V \rho$, we need to choose a Gaussian surface S. We choose a sphere around the charge, since this makes the integration easy. The right-hand side is

$$\int_V \rho = Q \tag{1.15}$$

where V is the volume inside the sphere. The left-hand side is

$$
\begin{aligned}
\int_S D &= \int_0^{2\pi} \int_0^{\pi} D_0 r^2 \sin\theta\, d\theta\, d\phi \\
&= 4\pi r^2 D_0.
\end{aligned}
$$

By Gauss's law, we know that $4\pi r^2 D_0 = Q$. Solving for D_0 and substituting into (1.14), we obtain

$$D = \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi \tag{1.16}$$

for the electric flux density due to the point charge. Note that 4π is the total amount of solid angle for a sphere and $\sin\theta\, d\theta\, d\phi$ is the differential element of solid angle, so this expression simply states that the amount of flux per solid angle is the same for any distance from the charge.

Line Charge

For a line charge with charge density ρ_l C/m, the tubes of flux extend out radially from the line, as shown in Fig. 1.22. These are tubes in $d\phi\, dz$ in the cylindrical coordinate system. In order to start with standard tubes of the same density everywhere in space, we add a factor ρ, so that D has the form

$$D = D_0 \rho\, d\phi\, dz \tag{1.17}$$
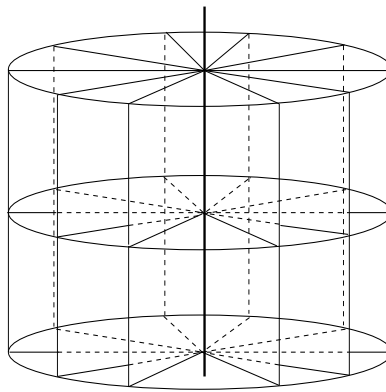
where we need to find D_0.

Figure 1.22: Electric flux density due to a line charge. Tubes of D extend radially away from the vertical line of charge.

The most convenient Gaussian surface to use in applying Gauss's law is a cylinder of height b centered on the line charge. The right-hand side of Gauss's law becomes

$$\int_V \rho = \int_0^b \rho_l \, dz$$
$$= b\rho_l.$$

The left-hand side is

$$\int_S D = \int_0^b \int_0^{2\pi} D_0 \rho \, d\phi \, dz$$
$$= 2\pi b \rho D_0.$$

By Gauss's law, we know that $2\pi b \rho D_0 = b\rho_l$. Solving for D_0 and substituting into (1.14), we obtain

$$D = \frac{\rho_l}{2\pi} \, d\phi \, dz \tag{1.18}$$

for the electric flux density due to the line charge.

Plane Charge

The tubes of flux from a plane charge with density ρ_s C/m^2 extend out perpendicular to the plane (Fig. 1.23). If the charge lies on the y–z plane, then D is a multiple of $\star \, dx = dy \, dz$. For $x > 0$ the tubes point in the $+x$ direction, and for $x < 0$ the tubes point in the $-x$ direction, so that

$$D = \begin{cases} D_0 \, dy \, dz & x > 0 \\ -D_0 \, dy \, dz & x < 0 \end{cases} \tag{1.19}$$

where we must find D_0.

To apply Gauss's law, we choose as the Gaussian surface the faces of a cube which is centered on the plane of charge. The right–hand side of Gauss's law becomes

$$\int_V \rho = \int_0^a \int_0^a \rho_s \, dy \, dz$$
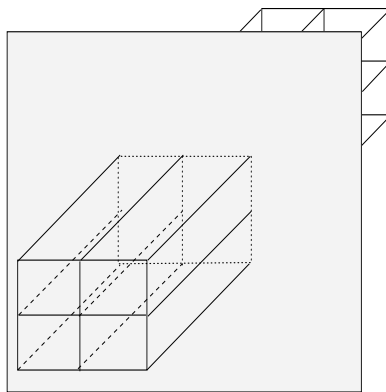$$= a^2 \rho_s$$

Figure 1.23: Electric flux density due to a plane charge. Tubes of D extend away from both sides of the plane. Only a few of the tubes are shown.

where a is the length of the sides of the cube. For the left-hand side, we only need to integrate over the front and back sides of the cube, since none of the tubes of D cut through the other sides of the cube.

$$
\begin{aligned}
\int_S D &= \int_0^a \int_0^a D_0 \, dy \, dz - \int_0^a \int_a^0 D_0 \, dy \, dz \\
&= 2a^2 D_0
\end{aligned}
$$

where we have switched the limits on the $-x$ face of the cube because its orientation is opposite that of the $+x$ face. By Gauss's law, we know that $2a^2 D_0 = a^2 \rho_s$. Solving for D_0 and substituting into (1.14), we obtain

$$
D = \begin{cases} \frac{\rho_s}{2} \, dy \, dz & x > 0 \\[2mm] -\frac{\rho_s}{2} \, dy \, dz & x < 0 \end{cases} \tag{1.20}
$$

for the electric flux density due to the plane charge.

1.5   Hodge Star Operator

One of the main points of this chapter is that while the electric field is a single physical quantity, we employ different mathematical representations in order to emphasize and work with different aspects of the field. Since the electric field intensity E and the electric flux density D represent the same field, it is clear that there must be some relationship between E and D. This relationship is expressed using the Hodge star operator. The star operator works by taking a form and converting it to a new form with the "missing" differentials. Here is how the star operator acts on 1-forms and 2-forms:

$$
\begin{aligned}
\star \, dx &= dy \, dz, & \star \, dy \, dz &= dx \\
\star \, dy &= dz \, dx, & \star \, dz \, dx &= dy \\
\star \, dz &= dx \, dy, & \star \, dx \, dy &= dz
\end{aligned}
$$

Note that the vector dual to a 1-form a is the same as the vector dual to the 2-form $\star a$. Also, the star operator applied twice is the identity, so that $\star \star \alpha = \alpha$ for any form α.

Consider the 1-form dx. The surfaces of dx are perpendicular to the x direction. Applying the star operator gives $dy \, dz$, which has tubes in the x direction, so that the surfaces of dx are perpendicular to the tubes of $\star \, dx$. This is true in general: the star operator on 1-forms and 2-forms always makes surfaces into perpendicular tubes and tubes into perpendicular surfaces. This is illustrated in Fig. 1.24.

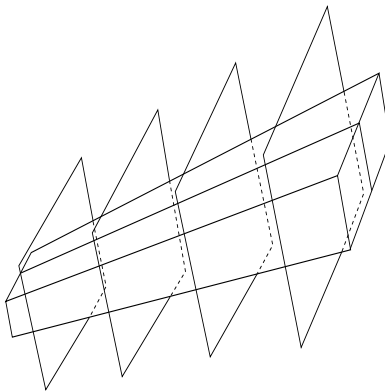**Example 1.6.** The Star Operator applied to a 1-form.

Figure 1.24: The star operator takes 1-form surfaces and makes them into perpendicular 2-form tubes.

If $E = 3\,dx + 5\,dy$, then

$$
\begin{aligned}
\star E &= 3\star dx + 5 \star dy \\
&= 3\,dy\,dz + 5\,dz\,dx.
\end{aligned}
$$

Note that $\star(3\,dy\,dz + 5\,dz\,dx) = 3\,dx + 5\,dy$.

In curvilinear coordinates, differentials must be converted to unit differentials before the star operator is applied. The star operator in cylindrical coordinates acts as follows:

$$
\begin{aligned}
\star\,d\rho &= \rho\,d\phi \wedge dz \\
\star\,\rho\,d\phi &= dz \wedge d\rho \\
\star\,dz &= d\rho \wedge \rho\,d\phi
\end{aligned}
$$

so that ρ must be included with $d\phi$ before the star operator can be applied to make it a 2-form with unit differentials. Also, $\star 1 = \rho\,d\rho\,d\phi\,dz$. As with the rectangular coordinate system, $\star\star = 1$, so that this same table can be used to convert 2-forms to 1-forms.

As in the cylindrical coordinate system, differentials of the spherical coordinate system must be converted to length elements before the star operator is applied. The star operator acts on 1-forms and 2-forms as follows:

$$
\begin{aligned}
\star\,dr &= r\,d\theta \wedge r\sin\theta\,d\phi \\
\star\,r\,d\theta &= r\sin\theta\,d\phi \wedge dr \\
\star\,r\sin\theta\,d\phi &= dr \wedge r\,d\theta
\end{aligned}
$$

Again, $\star\star = 1$, so that this table can be used to convert 2-forms to 1-forms. The star operator applied to one is $r^2\sin\theta\,dr\,d\theta\,d\phi$.

1.6 Electric Field Constitutive Relation

As noted in the introduction to this chapter and in the previous section, the electric field intensity 1-form E and the electric flux density D are two different representations of the same physical quantity. The mathematical relationship between E and D is known as a constitutive relation. The constitutive relation is written using the Hodge star operator as

$$
D = \epsilon_0 \star E
$$

where ϵ is a constant known as the permittivity of a medium. In a vacuum, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m³. The electric charges in materials interact with electric fields in an extremely complicated manner, but as will be discussed later, the average effect of these interactions can often be modelled by changing the value of the permittivity. This equation shows that surfaces of E yield perpendicular tubes of D (see Fig. 1.24). The constant scales the sizes of the tubes without changing their direction.

1.7   Electric Field Energy

An electric field contains a certain amount of potential energy stored in the space in which the field exists. This energy is represented by a 3-form, and the boxes of this 3-form are produced by the union of electric field intensity surfaces and flux density tubes. The electric field energy density 3-form is

$$w_e = \frac{1}{2} E \wedge D \tag{1.21}$$

where the factor of $1/2$ arises from the way the electric field has been defined. (If two charges are near each other, stored potential energy can be converted to kinetic energy by allowing one charge to accelerate away. The energy of the field due to the second charge remains but cannot be extracted, so we exclude it from the definition of w_e.)

1.7.1   Capacitance

Two nearby conductors that are oppositely charged allow energy to be stored in the electric field between them. The energy stored in the field can be increased or decreased by adding or taking away from the charges on the conductors. A measure of the amount of energy that can be stored by such a capacitor is the capacitance, defined by

$$C = \frac{Q}{V} \tag{1.22}$$

where Q is the charge on one of the plates and V is the voltage between the plates. The units are Farads, or Coloumbs per Volt.

Assuming a charge Q stored by a capacitor, the electric flux density D can be computed using Gauss's law. The constitutive relation $D = \epsilon \star E$ then gives E, from which V can be found by integrating E along a path from one of the conductors to the other. In principle, the capacitance of any pair of conductors can be found using this method.

Example 1.7. Parallel Plate Capacitor

Consider two parallel plates of area A, separated by a distance d. If one of the plates has charge Q and the other $-Q$, we need only find the voltage between them to find the capacitance. From the previous section, the electric flux between the plates is

$$D = \frac{Q}{A} \, dx \, dy$$

where the plates are perpendicular to the z-axis. Using the constitutive relation $D = \epsilon \star E$, we find $E = Q/(\epsilon A) \, dz$. The voltage between the plates is

$$\begin{aligned} V &= \int_0^d \frac{Q}{\epsilon A} \, dz \\ &= \frac{dQ}{\epsilon A} \end{aligned}$$

By Eq. (1.22), the capacitance of the plates is

$$C = \frac{\epsilon A}{d}. \tag{1.23}$$

1.8 Exercises

1.1. Draw the 1-forms (a) dy, (b) $dx + dy$ and (c) $x\,dx$.

1.2. (a) Find the vectors corresponding to the 1-forms in Ex. 1.8. (b) Draw the vectors.

1.3. Draw the 2-forms (a) $dy \wedge dz$, (b) $z\,dy \wedge dz$ and (c) $dy \wedge dz + dz \wedge dx$.

1.4. (a) Find the vectors corresponding to the 2-forms in Ex. 1.8. (b) Draw the vectors.

1.5. Let $a = 5\,dx + dz$, $b = 2dy$ and $c = 3\,dx \wedge dz$. (a) Find $a \wedge b$ and $a \wedge c$. (b) Find the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ corresponding to a, b and c. (c) Compute $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c}$. (d) Convert $\mathbf{a} \times \mathbf{b}$ into a 2-form.

1.6. Find the exterior product of (a) $3\,dx \wedge dy - xy\,dz \wedge dx$ and $z\,dy \wedge dz + 2x\,dz \wedge dx$ and (b) $dx + y\,dy$ and $xy\,dx \wedge dy \wedge dz$.

1.7. Integrate the 1-form $x\,dx + dy$ over the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

1.8. Draw pictures for $\star\,dx$ and dx. (a) What is the direction of the tubes of $\star\,dx$? (b) How does this relate to the surfaces of dx?

1.9. (a) Find the vectors corresponding to $dx + dz$ and $x\,dy\,dz + dy\,dx$. (b) Apply the star operator to these forms. (c) Find the vectors corresponding to your answers to part b.

1.10. Apply the star operator to (a) $4\,dx + x^2\,dy$, (b) $3\,dx\,dz + 4\,dy\,dx$, (c) $f\,dy\,dz - g\,dy\,dx$.

1.11. Find the vectors corresponding to (a) $d\rho + \rho\,d\phi + dz$, (b) $d\phi$, (c) $\rho\,dz + \frac{1}{\rho}\,d\phi$.

1.12. Find the vectors corresponding to (a) $\rho\,d\phi\,dz + dz\,d\rho + \rho\,d\rho\,d\phi$, (b) $4\,d\phi\,dz$, (c) $g\rho\,dz\,d\rho + h\frac{1}{\rho}\,d\phi\,dz$.

1.13. Find the 1-forms corresponding to (a) $\hat{\rho} + \hat{\phi} + \hat{z}$, (b) $\rho\hat{\phi}$, (c) $\rho\hat{\rho} + \frac{1}{\rho}\hat{\phi}$.

1.14. Find the 2-forms corresponding to the vectors in Ex. 1.8.

1.15. Find the vectors corresponding to (a) $dr + r\,d\theta + r\sin\theta\,d\phi$, (b) $d\phi$, (c) $r\,dr + \frac{1}{r}\,d\phi$.

1.16. Find the vectors corresponding to (a) $r\,d\phi\,dr + d\theta\,dr + r\,dr\,d\phi$, (b) $4\,d\phi\,d\theta$, (c) $gr\,d\theta\,dr + h\frac{1}{r}\,d\phi\,d\theta$.

1.17. Find the 1-forms corresponding to (a) $\hat{r} + \hat{\phi} + \hat{\theta}$, (b) $r\hat{\phi}$, (c) $r\hat{r} + \frac{1}{r}\hat{\theta}$.

1.18. Find the 2-forms corresponding to the vectors in Ex. 1.8.

1.19. (a) Apply the star operator to the forms in Ex. 1.8. (b) Apply the star operator to the forms in Ex. 1.8. (c) Apply the star operator to the forms in Ex. 1.8. (d) Apply the star operator to the forms in Ex. 1.8.

1.20. Apply the star operator to the forms in Ex. 1.8. Compare to the forms in Ex. 1.8. Repeat for Ex. 1.8 and 1.8.

1.21. Find the electric field intensity E due to two point charges a distance d m apart. (Hint: do a change of variables to transform the electric field 1-form for one of the charges to the coordinate system with origin at the other charge.)

1.22. Find the electric flux density D inside and outside a conducting sphere of radius 3 m with surface charge density 2 C/m^2.

1.23. Find the capacitance of two concentric conducting cylinders, with radii a and b m such that $a < b$ and lengths l m.

Chapter 2

MAGNETOSTATICS

2.1 Introduction

The previous chapter was designed to familiarize the reader with the important properties of eletric fields. Electric fields were introduced using a flat panel display example. Similarly, this chapter introduces the important properties of magnetic fields using a current technology, the "write head" on a computer memory disk. In some ways magnetic fields are more intuitive than electric fields because since childhood most of us have experimented with magnets. We have a natural feel for the attraction of opposite poles of a magnet. Most have seen the reaction of iron shavings to the magnetic field surrounding a magnet. Despite this familiarity, many of the properties of magnetic fields are still a bit mysterious. This chapter removes the shroud of mystery by describing magnetic fields in both mathmatical and physical language. As in the previous chapter, this chapter uses differential forms as a vehicle to connect the algebra and geometry of magnetic fields.
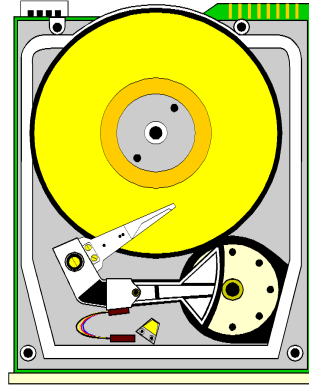

Figure 2.1: Magnetic disk drive.

Let us begin our description fields by considering the magnetic disk drive shown in Fig. 2.1. It is typical of drives used for storage in most current computer memory applications. Magnetic storage technology is the industry standard because of its high density and low cost when compared to solid state storage alternatives. This device consists of a rotating disk coated with a magnetic material. Bits of information are encoded in small areas (< 1 square micron) of the magnetic material by changing its magnetic alignment. The magnetic alignment is detected by the "read head" and the alignment can be changed with the "write head." Although the precise details of how the read and write heads interact with the disk to either retrieve or store data within the magnetic materials is beyond the scope of this chapter, a general description of the process is presented in what follows.

Figure 2.2 shows a two-dimensional view of a write head above a rotating disk. The write head is shown as consisting of two opposing magnetic poles 'flying' at a small distance above the magnetic material coated on the disk. As the head flies over a section of the disk, the magnetic material (which can be thought of as microscopic magnets
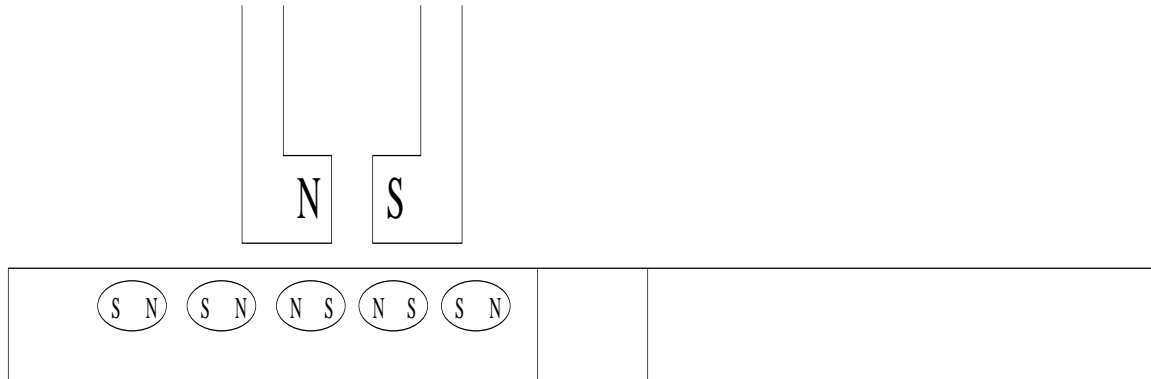
Figure 2.2: Disk drive write head.

embedded in a viscous material) orient themselves as their poles are attracted to the poles of the magnet as shown in the figure. The direction of orientation of the magnetic material depends on the direction of the magnet. If the polarity of the magnet is changed, the alignment of the magnetic material follows. For computer applications, one magnetic orientation is interpreted as a "one" bit and the other as a "zero" by the read head.
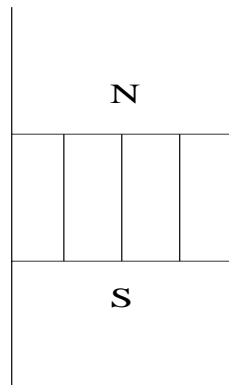

Figure 2.3: Gap between north and south poles of a magnet.

Although the force of attraction of the magnet extends beyond the 'gap' region between the two poles of the magnet, for simplicity let us restrict ourselves to the behavior in the gap between the two poles of a magnet for the time being. Figure 2.3 depicts the gap between two opposite poles of a magnet. We note the similarity between this structure and the parallel plate capacitor. In the case of the capacitor we defined the electric field and flux between the plates of the capacitor in terms of the voltage difference between the plates and the charge on the plates. In the magnetic case let us begin by describing the flux between the two opposite poles of the magnet. As in the case of the parallel plate capacitor we can think of flux as a connection in space between two equal and opposite charges. In the magnetic problem the charges are replaced by magnetic poles. The unit associated with the magnetic poles is the Weber, a unit of magnetic charge. (To date an isolated magnetic charge, or magnetic monopole, has not been discovered. For every north pole one finds an equal and opposite south pole, the two being inseparable.) As in the case of electric flux we can think of tubes of magnetic flux connecting the north and south poles of the magnet as shown in the figure. To find the amount of flux between the poles one only needs to count the number of tubes. In general this may be computed using the following integral:

$$\Psi = \int \int B_z \, dx \, dy$$

where $B_z \, dx \, dy$ is the 2-form representing the flux tubes. The 2-form coefficient, B_z, represents the amount of flux per unit area and is called the magnetic flux density. A high magnetic flux density means a strong magnetic attraction exists between the two poles of the magnet, and the flux tubes are smaller and more densely packed. A large electromagnet generates a magnetic flux density of several Webers/m^2 or equivalently, several Teslas.
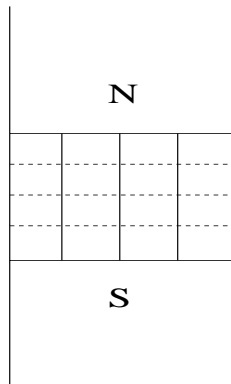

Figure 2.4: Relationship between magnetic flux tubes and perpendicular field intensity surfaces (dashed).

The reader will recall the relationship between electric field and electric flux. There is a similar relationship in magnetism. Recall that the electric field is represented as planes that are perpendicular to the tubes of electric flux, similarly the magnetic field may be represented as planes prependicular to the magnetic flux density. Figure 2.4 shows the relationship between the magnetic field and magnetic flux between the poles of a magnet. The magnetic flux is a 2-form and the magnetic field is a 1-form. The magnetic field and flux are related by the constitutive relation:

$$B = \mu \star H$$

Here μ is the permeability of the medium, and is $\mu_0 = 4\pi \times 10^{-6}$ Henrys in free space. It is a measure of the degree to which a material responds to a magnetic field. Performing a hodge star operation on both sides of the equation gives

$$H = \star B / \mu$$

Geometrically this is interpreted as the 2-form $B \, dx \, dy$ tubes yielding perpendicular 1-form magnetic field planes, $H \, dz$. The units of the magnetic field are A/m.

The magnetic field and flux can be combined to find the magnetic energy stored between the poles of the magnet. The exterior product of the two quantities is

$$w_m = \frac{1}{2} H \wedge B.$$

This 3-form represents boxes of energy. Integrating the boxes over a region gives

$$E_m = \int_R w_m$$

which computes the magnetic energy stored in a region of space. A greater density of boxes means that there is a greater amount of energy stored in space per unit volume. The capacity to store energy in the magnetic field is measured in terms of the inductance of the medium. We write this for the example described above as

$$L = B/H \text{ Webers/Ampere}$$

where the units are defined as Henrys.

2.2   Gauss's Law for the Magnetic Field

2.2.1   Magnetic Flux Density

As we saw in the introduction, the magnetic flux density is similar to the electric flux density, and is represented by a 2-form,

$$B = B_1 \, dy \, dz + B_2 \, dz \, dx + B_3 \, dx \, dy. \tag{2.1}$$

The coefficients of B have units Wb/m^2, and B has units of Wb, which is the unit of magnetic charge. The tubes of B represent flux extending from north magnetic poles to south magnetic poles. Magnetic flux tubes can also form closed loops around a current.
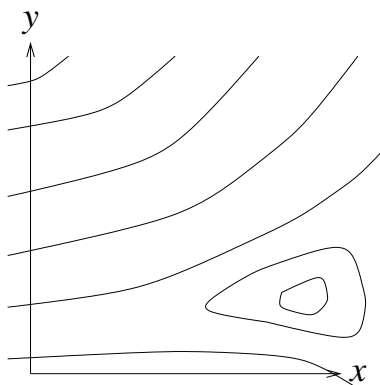
2.2.2   Gauss's Law

Figure 2.5: Tubes of magnetic flux must either extend to infinity or form closed loops.

Gauss's law for magnetic flux density is

$$\oint_S B = 0 \tag{2.2}$$

where S is a closed Gaussian surface. The right hand side of this equation is zero, since no isolated magnetic charges have ever been observed. Magnetic fields can only be created by moving charges or changing electric fields, and so magnetic sources can only exist in the form of pairs consisting of north and south poles. This law states that tubes of the 2-form B can never end—they must either form closed loops or go off to infinity. If the tubes of magnetic flux were to end, then if the Gaussian surface S contained the end point, a tube would pass into the closed surface which would not come back out, and the integral in Eq. (2.2) would be nonzero.

2.3   Magnetic Field Intensity

Just as the electric field can be described by both flux density and field intensity, we also define a field intensity 1-form

$$H = H_1 \, dx + H_2 \, dy + H_3 \, dz \tag{2.3}$$

for the magnetic field. The units of the coefficients are A/m, and H has units of Amperes. In a region containing no currents or time–varying fields, the surfaces of the magnetic field intensity 1-form H can be viewed as equipotentials for the magnetic potential.

2.4   Ampere's Law

Ampere's law for static fields is

$$\oint_P H = \int_A J \tag{2.4}$$

where A is a two–dimensional region, and P is the boundary of that region. P is always a closed loop and is known as the *Amperian contour*. It is similar to the Gaussian surface of Chap. 3. The 2-form J represents the density of current flow, and has units of A, so that one tube of J represents one Ampere of current. For a wire carrying current, the tubes of J are parallel to and inside of the conductor. Ampere's law states that the number of surfaces of H pierced by the Amperian contour is equal to the number of tubes of J passing through the loop. Graphically, this means that surfaces of H extend away from tubes of J, as shown in Fig. 2.6. We say that the tubes of J are sources for the surfaces of H.
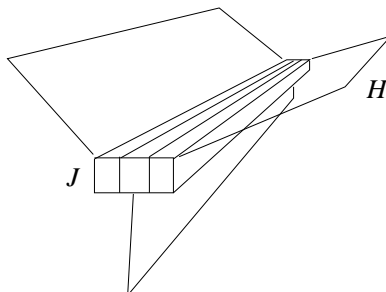

Figure 2.6: A graphical representation of Ampere's law for no time-varying electric flux: tubes of J produce surfaces of H.

The 1-form shown in Fig. 2.6 has the property that integrals around closed paths can be nonzero. In this case, the integral is nonzero if tubes of the current density J pass through the Amperian contour. This is an example of a nonconservative field. The surfaces of a nonconservative 1-form do not represent regions of a fixed, unique potential, but they do represent change in potential, since each time a magnetic charge moves across a surface, its potential changes by a unit value.
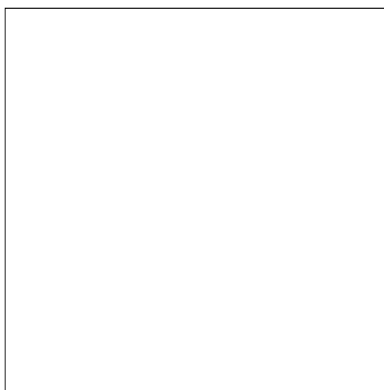

Figure 2.7: Just as the walkers move continually upward or downward as they travel around the closed staircase, a magnetic charge moving around a current-carrying wire experiences a continual change from high to low potential and gains energy, or in the opposite direction the potential changes from low to high and work must be done on the charge.

We will now employ this law find the magnetic field intensity for several choices for the current density 2-form J. We find the magnetic field for a current confined to a straight wire, a sheet current, and a solenoid.

2.4.1 Line Current

If a current I_l A is flowing along the z-axis, sheets of the H 1-form will extend out radially from the current, as shown in Fig. 2.8. These are the planes of $d\phi$ in the cylindrical coordinate system. We know that H is
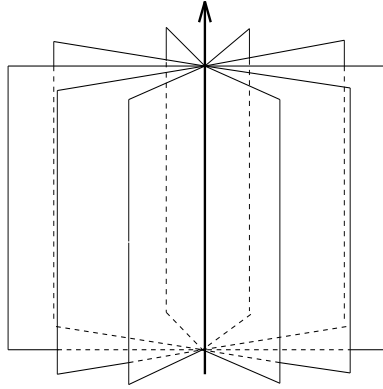
$$H = H_0 \, d\phi \tag{2.5}$$

Figure 2.8: Magnetic field intensity H due to a line current.

where H_0 is a constant we need to find using Ampere's law.

We choose the Amperian contour P to be a circle around the z-axis. Since we assume that $D = 0$, the right–hand side of Ampere's law becomes

$$\int_A J = I_l$$

since I_l A of current flows through the disk A that lies inside the circle P. The left-hand side is the integral of H over the circle,

$$\int_P H = \int_0^{2\pi} H_0 \, d\phi$$
$$= 2\pi H_0.$$

Solving for H_0 shows that $H_0 = \frac{I_l}{2\pi}$, so that

$$H = \frac{I_l}{2\pi} \, d\phi \tag{2.6}$$

is the magnetic field intensity due to the line current.

2.4.2   Sheet Current

For a sheet current of J_s A/m flowing in the $+z$ direction on the $x = 0$ plane, the surfaces of H will extend out from the current parallel to the $y = 0$ plane (Fig. 2.9). These are surfaces of dy. For $x < 0$, the surfaces of H must be oriented in the opposite direction as they are for $x > 0$, so we can say that

$$H = \begin{cases} H_0 \, dy & x > 0 \\[2mm] -H_0 \, dy & x < 0 \end{cases} \tag{2.7}$$

where H_0 is an unknown constant.

In this case, we will take the Amperaan contour to be a square of side b that is cut in half by the current sheet. The total current passing through the square is.

$$\int_A J = \int_0^b J_s \, dy$$
$$= b J_s.$$

The integral of H around the sides of the square is

$$\int_P H = \int_0^b H_0 \, dy - \int_b^0 H_0 \, dy$$
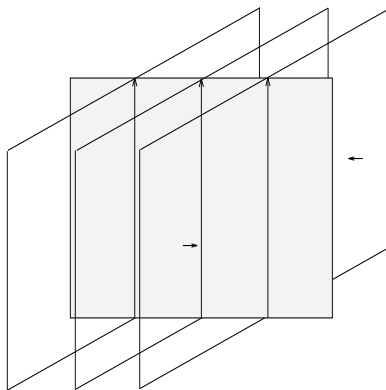$$= 2b H_0$$

Figure 2.9: Magnetic field intensity due to a sheet current. Surfaces of H extend out away from the sheet. The surfaces intersect the sheet along lines of current flow.

since the sides of the square parallel to the x axis do not pass through any of the surfaces of H. Solving for H_0 and substituting into the expression for H above gives

$$H = \begin{cases} \frac{J_s}{2}\,dy & x > 0 \\[2mm] -\frac{J_s}{2}\,dy & x < 0 \end{cases} \tag{2.8}$$

for the magnetic field intensity due to the current sheet.

2.4.3   Solenoid

Consider a solenoid of radius a with N turns of wire per meter. Surfaces of H extend out from each loop of wire in all radial directions, but cancel such that H has only a dz component, as in Fig. 2.10. H can be written $H_0\,dz$.
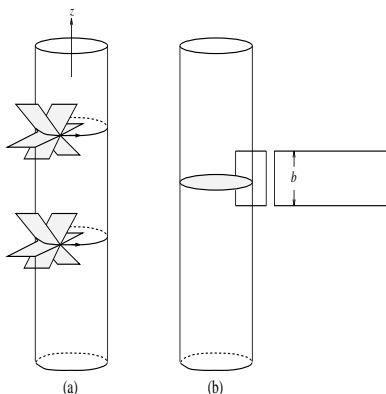
Figure 2.10: (a) Two of the turns of a solenoid. Surfaces of H extend out radially from the turns. (b) At a point between the two loops, by the symmetry of the solenoid all components of H cancel except for the dz component. Also shown are two Amperian contours used to compute H using Ampere's law.

Integrating $H_0\,dz$ over the Amperian contour in Fig. 2.10b that is outside the solenoid gives

$$\oint H = \int_0^b H_0\,dz$$
$$= bH_0$$

since the loop closes far away from the solenoid where the field has fallen to zero. The only contribution to the integral is along the vertical side near the solenoid. This loop encloses no current, so by Ampere's law, $bH_0 = 0$ and the magnetic field outside the solenoid must be zero.

Integrating $H_0\,dz$ over the other Amperian contour in Fig. 2.10b gives bH_0, since the integral is nonzero only along the vertical side of the loop that is inside the solenoid. The loop encloses a current of bNI A. By Ampere's law, $H_0 = NI$ and the magnetic field inside the solenoid is $NI\,dz$.

2.5   Magnetic Field Energy

Just as the electric field between the plates of a capacitor stores energy, a magnetic field stores energy as well. If one connects the leads of a coil with a few hundred turns of wire to a small battery, when the leads are disconnected quickly the energy contained in the magnetic field around the wire is dissipated by a spark as the field collapses. In this section, we give a mathematical definition for energy stored by the magnetic field, and use this to define inductance, which measures the amount of energy stored by a conductor of a given shape for a unit amount of current flow.

The energy stored by the magnetic field is

$$w_m = \frac{1}{2}H \wedge B. \tag{2.9}$$

Using the constitutive relation for the magnetic field, this can be rewritten as

$$w_m = \frac{1}{2}\mu H \wedge \star H. \tag{2.10}$$

The quantity w_m is a 3-form representing the density of energy stored, and has units J/m^4.

2.5.1   Inductance

As with a capacitor, a current–carrying wire allows energy to be stored in the magnetic field around the wire. A measure of the amount of energy stored is the inductance in Henrys per meter,

$$L = \frac{N\psi}{I} \tag{2.11}$$

where ψ is the flux linked by a closed loop, N is the number of turns of conductor around the loop and I is the current flowing in the conductor. ψ is given by the integral

$$\psi = \int_A B \tag{2.12}$$

where A is a disk bounded by the closed loop and B is the magnetic flux density produced by the current flow.

Example 2.1. Inductance of a solenoid.

As we found earlier in this chapter, inside a solenoid of radius a centered on the z–axis the magnetic field intensity is $H = NI\,dx$. Integrating B over a slice of the solenoid gives $\psi = \mu NI\pi a^2$, so that the inductance of the solenoid per unit length is

$$L = \pi a^2 \mu N^2.$$

2.6 Exercises

2.1. Find the inductance of pair of parallel wires of length 2 m, separated by a distance 1 cm. At one end, the wires are connected. At the other, a current source drives the system. Neglect effects to fringing of fields at the ends of the wires.

Chapter 3

TIME VARYING FIELDS

3.1   Introduction

In previous chapters, we have seen that electric fields are produced by static electric charge and magnetic fields are produced by moving charges, or currents. In this chapter, we will see that an electric field which changes in time produces a magnetic field. Similarly, a magnetic field which changes in time produces an electric field.

Historically, the experimental fact that time–varying magnetic fields produce an electric field was observed first, by Faraday and Henry in the 1830s, and is written mathematically as Faraday's law:

$$\oint_C E = -\frac{\partial}{\partial t}\int_A B \tag{3.1}$$

where C is a closed contour and A is a surface bounded by the contour. This law shows that surfaces of the electric field intensity 1-form E are produced by tubes of time–varying magnetic flux density.

Maxwell hypothesized that a similar relationship between time–varying electric fields and magnetic field existed. One way to understand this is to consider a capacitor driven by a sinusoidal voltage source. We can apply Ampere's law to this circuit, and place the Amperian contour so that it passes around one of the leads to the capacitor, as shown in Fig. 3.1.
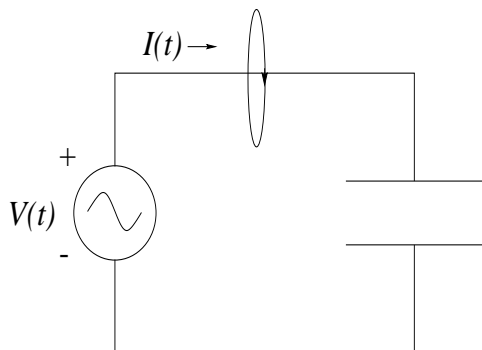

Figure 3.1: A sinusoidal voltage source drives a capacitor. An Amperian contour is shown passing around one of the connecting wires.

From the discussion of Ampere's law in Chap. 3, we have that

$$\oint_C H = \int_A J \tag{3.2}$$

where C is the contour passing around one of the connecting wires. If we choose A to be a flat disk in the plane of

the contour, then the right hand side is equal to I, the total current passing through the wire. The left–hand side is therefore equal to I as well.
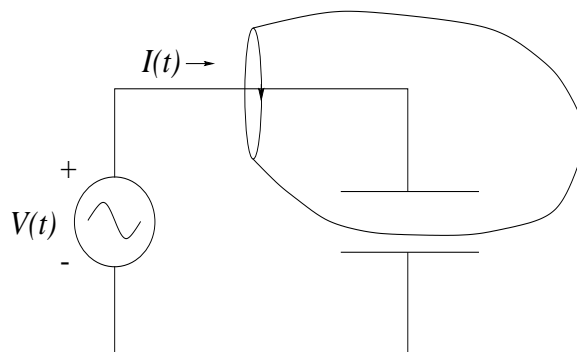

Figure 3.2: Ampere's law applied to the same contour as in Fig. 3.1 but with the surface passing between the capacitor plates. No electric current flows through the surface, so the electric field between the plates must contribute to the right–hand side of Ampere's law. This is *displacement current*.

Recall that Ampere's law holds for *any* surface A, as long as it is bounded by the Amperian contour. If we fix the contour and stretch the surface A, the left–hand side does not change. Suppose that we choose A to be a surface which passes between the plates of the capacitor, as in Fig. 3.2. Clearly, no current passes through this surface, since it is not intersected by the wire. Between the plates of the capacitor is an insulating material, so no current flows between the plates. We arrive at a contradiction: the left–hand side of (3.2) becomes zero, but the right–hand side is equal to I.

Maxwell postulated an additional term to be added to Eq. (3.2) which resolves this contradiction. Because of the charge stored in the capacitor, there is an electric field between the plates. This changing electric field leads to an effective current which appears in Ampere's law:

$$\oint_C H = \frac{\partial}{\partial t} \int_A D + \int_A J. \tag{3.3}$$

This new term is known as the *displacement current*.

Equations (3.1) and (3.3) are valid for time–varying fields, as are Gauss's laws for the electric and magnetic flux density discussed in the preceding chapters. We now have the full set of laws which describe the behavior of the electromagnetic field. In this chapter, we will examine in more detail Maxwell's laws for time–varying fields, as a preparation for treating in later chapters the most important application of these laws: electromagnetic waves.

3.2   Maxwell's Laws in Integral Form

In this section, we summarize the complete set of Maxwell's laws in integral form. Table 3.1 gives all of the field and source quantities, the degrees of the differential forms used to represent them, and their units. Maxwell's laws are

$$\oint_C E \;=\; -\frac{d}{dt} \int_A B$$
$$\oint_C H \;=\; \frac{d}{dt} \int_A D + \int_A J$$

$$\oint_S D = \int_V \rho$$

$$\oint_S B = 0 \tag{3.4}$$

where C is any closed contour, A is any surface bounded by the contour C, S is any closed surface, and V is the volume bounded by that surface. The first pair are Faraday's and Ampere's law, respectively, and the second pair are Gauss's laws for the magnetic and electric flux densities.

Quantity	Form	Type	Units	Vector/Scalar
Electric Field Intensity	E	1-form	V	**E**
Magnetic Field Intensity	H	1-form	A	**H**
Electric Flux Density	D	2-form	C	**D**
Magnetic Flux Density	B	2-form	Wb	**B**
Electric Current Density	J	2-form	A	**J**
Electric Charge Density	ρ	3-form	C	q

Table 3.1: The differential forms of electromagnetics, their degree, and the corresponding vector quantities.

In previous chapters, we have discussed Gauss's law for the electric and magnetic fields, and Ampere's law without the displacement current term. In the remainder of this section, we will examine Faraday's law in detail.

3.2.1 Faraday's Law

Faraday's law states that any closed contour which passes around tubes of time–varying magnetic flux density must pierce surfaces of the electric field intensity 1-form E. The graphical representation of this law is identical to that of Ampere's law in the previous chapter: just as tubes of current flow are sources for magnetic field surfaces, tubes of time–varying flux are sources for electric field surfaces.

Faraday's law is the basis for the electric generator. If a magnet and a loop of wire are rotated with respect to each other, then the magnetic flux through the coil of wire changes in time. By Faraday's law, the integral of the electric field intensity around the loop of wire is nonzero, and this potential causes current to flow. The electric transformer also uses the same principle: a primary coil produces a changing magnetic field, and the resulting tubes of flux pass through a secondary coil, and a voltage around the secondary is induced. Below are illustrations of a very simple transformer and an electric generator.

Example 3.1. Loop near time–varying current

A square loop of wire lies near a sinusoidal current $I(t) = \sin(100t)$ V flowing along the z axis (Fig. 3.3). From Ampere's law, the magnetic field produced by the current is $B = \mu I / (2\pi\rho) \, dz \, d\rho$. By Faraday's law, the voltage around the loop is

$$\oint E = -\int_a^b \int_0^c \frac{50}{\pi\rho} \cos(100t) \, dz \, d\rho$$

$$= \frac{50c}{\pi} \cos(100t) \ln\frac{b}{a} \text{ V}$$

Note that the induced voltage is independent of the value of the resistance in the loop.

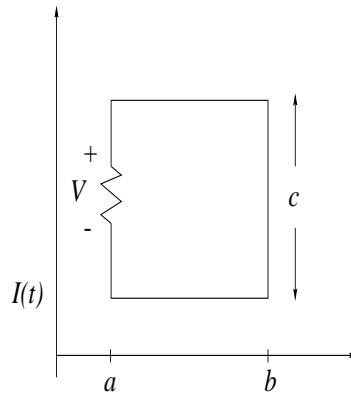**Example 3.2.** Simple electric generator

Figure 3.3: Faraday's law shows that a voltage is induced around a square conducting loop through which passes the magnetic field produced by a time–varying current.

In this example, we consider a very simple type of electric generator. A cylindrical conducting bar lies free to move on two conducting rails, as shown in Fig. 3.4. A constant magnetic field $B_0 \, dx \, dy$ passes between the rails. If the bar is moved at a constant velocity v, the voltage induced around the loop formed by the rails and bar is

$$
\begin{aligned}
\oint E &= -\frac{\partial}{\partial t} \int_0^{vt} \int_0^a B_0 \, dx \, dy \\
&= -\frac{\partial}{\partial t} vta B_0 \\
&= -va B_0 \text{ V}.
\end{aligned}
$$

Work required to move the bar is converted into electrical energy.
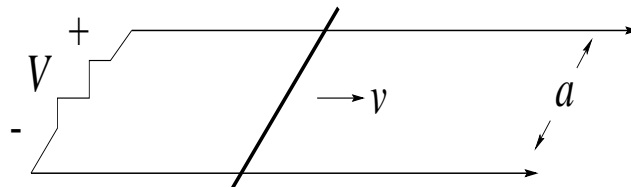

Figure 3.4: Simple electric generator. A conducting rod rolls on two rails, so that the magnetic flux through the circuit changes and a voltage is induced across the resistor.

3.3 Exterior Derivative

In this section, we define the exterior derivative operator, which will allow us to express Maxwell's laws as differential equations. This operator has the symbol d, and acts on a p-form to produce a new form with degree $p + 1$. This $(p + 1)$-form characterizes the spatial variation of the p-form.

The exterior derivative operator can be written as

$$
d = \left(\frac{\partial}{\partial x} \, dx + \frac{\partial}{\partial y} \, dy + \frac{\partial}{\partial z} \, dz \right) \wedge \tag{3.5}
$$

The d operator is similar to a 1-form, except that the coefficients are partial derivative operators instead of functions. When this operator is applied to a differential form, the derivatives act on the coefficients of the form, and the differentials combine with those of the form according to the properties of the exterior product.

Computing exterior derivatives is straightforward. One takes the partial derivative of a differential form by x and adds the differential dx from the left, repeats for x and y, and adds the three results. This process is very similar to implicit differentiation, except that one must combine new differentials with existing differentials using the exterior product. When taking the exterior derivative of a 1-form or 2-form, some terms may drop out due to repeated differentials.

Example 3.3. Exterior derivative of a 0-form.

The exterior derivative of $3x^2z$ is

$$
\begin{aligned}
d3x^2z &= (\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy + \frac{\partial}{\partial z}\,dz) \wedge 3x^2z \\
&= \frac{\partial}{\partial x}3x^2z\,dx + \frac{\partial}{\partial y}3x^2z\,dy + \frac{\partial}{\partial z}3x^2z\,dz \\
&= 6xz\,dx + 0 + 3x^2\,dz \\
&= 6xz\,dx + 3x^2\,dz
\end{aligned}
$$

The gradient of $3x^2z$ is $6xz\hat{x} + 3x^2\hat{z}$, so the exterior derivative of a 0-form is analogous to the vector gradient operator.

Example 3.4. Exterior derivative of a 1-form.

The derivative of the 1-form $f\,dx$ is

$$
\begin{aligned}
df\,dx &= (\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy + \frac{\partial}{\partial z}\,dz) \wedge f\,dx \\
&= \frac{\partial}{\partial x}f\,dx \wedge dx + \frac{\partial}{\partial y}f\,dy \wedge dx + \frac{\partial}{\partial z}f\,dz \wedge dx \\
&= \frac{\partial f}{\partial z}\,dz \wedge dx - \frac{\partial f}{\partial y}\,dx \wedge dy.
\end{aligned}
$$

The vector curl operator applied to $f\hat{x}$ yields $\frac{\partial f}{\partial z}\hat{y} - \frac{\partial f}{\partial y}\hat{z}$, which is the dual vector of the 2-form found above.

Example 3.5. Exterior derivative of a 2-form.

The derivative of the 2-form $3x\,dy \wedge dz - xy\,dz \wedge dx$ is

$$
\begin{aligned}
d(3x\,dy \wedge dz - xy\,dz \wedge dx) &= (\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy + \frac{\partial}{\partial z}\,dz) \wedge (3x\,dy \wedge dz - xy\,dz \wedge dx) \\
&= \frac{\partial}{\partial x}3x\,dx \wedge dy \wedge dz - \frac{\partial}{\partial y}xy\,dy \wedge dz \wedge dx \\
&= (3 - x)\,dx \wedge dy \wedge dz.
\end{aligned}
$$

The vector divergence of $3x\hat{x} - xy\hat{y}$ is $3 - x$, which is the coefficient of the 3-form found above. The exterior derivative of any 3-form would be a 4-form, which must be zero due to repeated differentials.

An important identity is

$$dd = 0 \tag{3.6}$$

meaning that the exterior derivative applied twice always yields zero. This relationship is equivalent to the vector identities $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times \mathbf{v}) = 0$. The exterior derivative also satisfies a product rule analogous to the product rule for the partial derivative,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{3.7}$$

where p is the degree of α.

3.3.1   Exterior Derivative in Curvilinear Coordinates

Regardless of the particular coordinate system, the form of the exterior derivative operator remains the same. If the coordinates are (u, v, w), the exterior derivative operator is

$$d = \left(\frac{\partial}{\partial u}\, du + \frac{\partial}{\partial v}\, dv + \frac{\partial}{\partial w}\, dw \right) \wedge \tag{3.8}$$

In cylindrical coordinates,

$$d = \left(\frac{\partial}{\partial \rho}\, d\rho + \frac{\partial}{\partial \phi}\, d\phi + \frac{\partial}{\partial z}\, dz \right) \wedge \tag{3.9}$$

which is the same as for rectangular coordinates but with the coordinates ρ, ϕ, z in the place of x, y, z. Note that the factor ρ associated with $d\phi$ must be present when converting to vectors or applying the star operator, but is not found in the exterior derivative operator.

The exterior derivative in spherical coordinates is

$$d = \left(\frac{\partial}{\partial r}\, dr + \frac{\partial}{\partial \theta}\, d\theta + \frac{\partial}{\partial \phi}\, d\phi \right) \wedge . \tag{3.10}$$

From these expressions, we see that computing exterior derivatives in a curvilinear coordinate system is no different from computing in rectangular coordinates.

3.4   Stokes' Theorem

The most important property of the exterior derivative is the generalized Stokes' theorem. If ω is a differential form, then

$$\boxed{\int_M d\omega = \oint_{bd\,M} \omega.} \tag{3.11}$$

M is some region of space, and $bd\,M$ is its boundary. The dimension of $bd\,M$ has to match the degree of ω. This formula may seem obscure because it is in abstract language, but the idea behind it is quite simple, especially when interpreted graphically.

If ω is a 0-form, then Stokes' theorem states that $\int_a^b df = f(b) - f(a)$. This is the fundamental theorem of calculus.

If ω is a 1-form, then $bd\,M$ has to be a closed path. M is a surface that has the path as its boundary. Graphically, Stokes' theorem says that the number of surfaces of ω pierced by the path is equal to the number of tubes of the 2-form $d\omega$ that pass through the path (Fig. 3.5).

If ω is a 2-form, then $bd\,M$ is a closed surface and M is the volume inside it. Stokes' theorem requires that the number of tubes of ω that cross the surface is equal to the number of boxes of $d\omega$ inside the surface, as shown in Fig. 3.6.

3.5   Maxwell's Laws in Point Form

For problems other than those that are very symmetric, such as point and line sources, Maxwell's laws in integral form are difficult to use analytically. A differential formulation is more appropriate in many cases, especially for the study of waves, which we will take up in the next chapter. In this section, we employ Stokes' theorem to convert Maxwell's laws from integral form to point form.
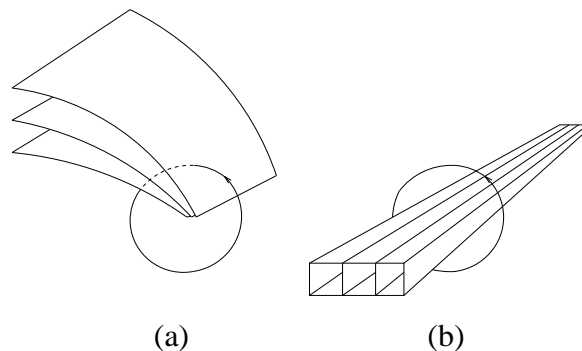
Figure 3.5: Stokes theorem for ω a 1-form. (a) The loop $bd\,M$ pierces three of the surfaces of ω. (b) Three tubes of $d\omega$ pass through a surfaces M bounded by the loop $bd\,M$.
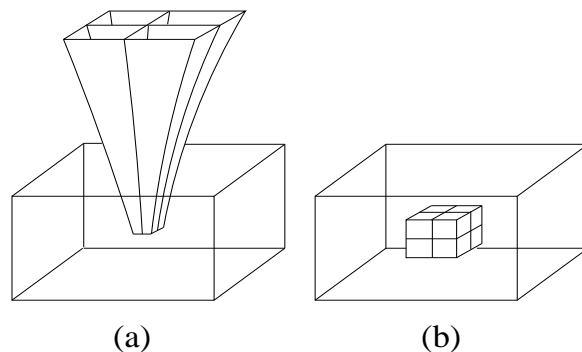

Figure 3.6: Stokes theorem for ω a 2-form. (a) Four tubes of ω pass through the closed, rectangular surface $bd\,M$. (b) Four boxes of the 3-form $d\omega$ lie inside the surface.

3.5.1 Faraday's and Ampere's Laws

Faraday's law in integral form is

$$\oint_P E = \int_A -\frac{\partial}{\partial t} B \tag{3.12}$$

We wish to apply Stokes' theorem to the left hand side of Faraday's law. Stokes' theorem for the case of a 1-form relates the integral of the 1-form over a closed path to the integral of the exterior derivative of the 1-form over a surface bounded by the path. This will lead to integrals over A on both sides of the equation, and since A is arbitrary we will then be able to remove the integrals and arrive at the desired differential equation.

By using Stokes' theorem (3.11) with M as the surface A and ω as the electric field intensity E, we have that

$$\oint_P E = \int_A dE \tag{3.13}$$

Substituting this relationship into Faraday's law gives

$$\int_A dE = \int_A -\frac{\partial}{\partial t} B \tag{3.14}$$

In general, even though two integrals are equal, the integrands may be different. In this case, however, the surface A is arbitrary, so the integrands of Eq. (3.14) must be equal. One way to see this is to shrink the surface A until it is very small. The integrals are then approximately equal to their arguments evaluated at a point inside A multipled by the area of A. The areas divide out in the limit, and the integrands are equal. We thus arrive at

$$\boxed{dE = -\frac{\partial}{\partial t} B} \tag{3.15}$$

This is Faraday's law in point form. Graphically, this law states that surfaces of E can only end along tubes of time-varying magnetic flux.

Using a similar argument, Ampere's law becomes

$$\boxed{dH = \tfrac{\partial}{\partial t}D + J.}$$
(3.16)

Graphically, Ampere's law shows that surfaces of H can only end along tubes of time-varying electric flux or tubes of electric current.

Example 3.6. Faraday's law in point form

We wish to find the magnetic field intensity if the electric field is $E = \sin x \cos \omega t \, dy$. By Faraday's law,

$$
\begin{aligned}
\frac{\partial}{\partial t}B &= -d(\sin x \cos \omega t \, dx) \\
&= -\cos x \cos \omega t \, dx \, dy.
\end{aligned}
$$

Integrating this result in t and using the constitutive relation $H = \star B/\mu$ shows that the magnetic field intensity is equal to $-\cos x \sin \omega t/(\omega \mu)\, dz$, where we ignore the possibility of another term which is constant in time.

Example 3.7. Ampere's law in point form

Determine the value of ω for the field in the previous example. Using Ampere's law and $D = \epsilon \star E$, we have that

$$-d(\cos x \sin \omega t/(\omega \mu)\, dz) = \frac{\partial}{\partial t}\epsilon \star(\sin x \cos \omega t \, dy)$$
(3.17)

which leads to the relationship $\omega = \pm 1/\sqrt{\epsilon \mu}$.

3.5.2   Gauss's Laws

Gauss's law for the electric flux density is

$$\oint_S D = \int_V \rho$$
(3.18)

Using Stokes' theorem with M as the volume V, $bd\ M$ as the surface S, and ω as the 2-form D shows that

$$\oint_S D = \int_V dD.$$
(3.19)

Combining this relationship with Eq. (3.18) leads to

$$\int_V dD = \int_V \rho.$$
(3.20)

Since V is arbitrary, the integrands must be equal, so that we have the relationship

$$\boxed{dD = \rho.}$$
(3.21)

This is Gauss's law for the electric field. Graphically, it states that tubes of electric flux density can end only on electric charges.

Similarly, Gauss's law for the magnetic field is

$$\boxed{dB = 0.}$$

(3.22)

Graphically, this law shows that tubes of magnetic flux density never end. They must form closed loops or extend to infinity.

Example 3.8. Gauss's law in point form

Could the magnetic field intensity be equal to the 1-form $\sin y \, dy$? Since

$$
\begin{aligned}
d \star \sin y \, dy &= d \sin y \, dz \, dx \\
&= \cos y \, dx \, dy \, dz
\end{aligned}
$$

is nonzero, the corresponding flux density would not satisfy Gauss's law, so H cannot be equal to $\sin y \, dy$.

In this section, we have derived Maxwell's laws in point form:

$$
\begin{aligned}
dE &= -\frac{\partial}{\partial t} B \\
dH &= \frac{\partial}{\partial t} D + J \\
dD &= \rho \\
dB &= 0.
\end{aligned}
$$

Together with the constitutive relatons $D = \epsilon \star E$ and $B = \mu \star H$, we now have a set of coupled partial differential equations which describe the electromagnetic field. In later chapters, we will apply standard techniques of partial differential equation theory, such as separation of variables, to solve these equations and gain further understanding of the behavior of electromagnetic fields.

3.6 Boundary Conditions

If a magnetic field changes abruptly along some boundary surface, Maxwell's laws require that an electric current flow along the boundary to account for the step in field intensity. Similarly, Maxwell's laws restrict the possible discontinuity in the electric field at a boundary. In this section, we derive expressions for these boundary conditions on E, H, D, and B.

3.6.1 Field Intensity

In this section, we derive boundary conditions for the electric and magnetic field intensity 1-forms E and H. As in Fig. 3.7, we denote the magnetic field on one side of a boundary as H_1, and the field on the other side as H_2. If we choose an Amperian contour with one side just above and the other just below the boundary, as shown in Fig. 3.7, the left hand side of Ampere's law becomes

$$\oint_C H = \int_P (H_1 - H_2)$$

(3.23)

in the limit as the width of the contour goes to zero and the sides of the contour meet each other along a common path P on the boundary. The right hand side is equal to the surface current flowing across the path P, so that

$$\int_P (H_1 - H_2) = \int_P J_s$$
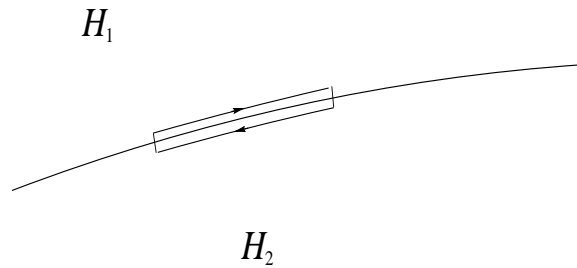
(3.24)

$$H_1$$

$$H_2$$

Figure 3.7: A discontinuity in the magnetic field above and below a boundary. The surface current flowing on the boundary can be found using an Amperian contour with infinitesimal width.

where J_s is a 1-form representing the surface current density on the boundary.

Since Eq. (3.24) holds for any path P on the boundary, the integrands must be equal on the boundary. We thus have that

$$J_s = (H_1 - H_2)|_B \tag{3.25}$$

where the right hand side is the restriction of the magnetic field discontinuity to the boundary. The 1-form J_s is represented graphically by the lines along which the 1-form $H_1 - H_2$ intersects the boundary, as shown in Fig. 3.8. Current flows along these lines. If the surfaces of $H_1 - H_2$ are parallel to the boundary, then the surfaces do not intersect, and the restriction is zero. Thus, Eq. (3.25) represents the tangential component of the magnetic field discontinuity. The direction of current flow along these lines can be obtained using the right hand rule: if the right hand is on the boundary and the fingers point in the direction of $H_1 - H_2$, then the thumb points in the direction of current flow.
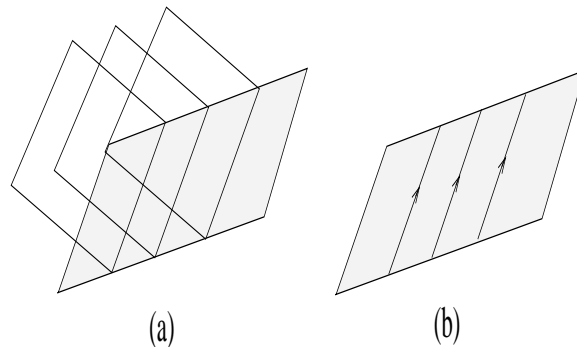

Figure 3.8: (a) The 1-form $H_1 - H_2$. (b) The 1-form J_s, represented by lines on the boundary. Current flows along the lines.

In order to compute the restriction mathematically, we employ an expression of the form $z = f(x, y)$ to represent the boundary, and replace all occurences of the variable z in $H_1 - H_2$ with the function $f(x, y)$, so that

$$
\begin{aligned}
J_s &= [H_1(x,y,z) - H_2(x,y,z)]|_{z=f(x,y)} \\
&= [H_{1x}(x,y,f) - H_{2x}(x,y,f)]\, dx + [H_{1y}(x,y,f) - H_{2y}(x,y,f)]\, dy + [H_{1z}(x,y,f) - H_{2z}(x,y,f)]df \\
&= \left[H_{1x} - H_{2x} + \frac{\partial f}{\partial x}(H_{1z} - H_{2z}) \right] dx + \left[H_{1y} - H_{2y} + \frac{\partial f}{\partial y}(H_{1z} - H_{2z}) \right] dy
\end{aligned}
$$

If part of the boundary is parallel to the $x-y$ plane, then the boundary must be expressed as $x = g(y, z)$ or $y = h(x, z)$. The following examples provide illustrations of the application of this boundary condition.

Example 3.9. Magnetic field intensity boundary condition

Suppose that the magnetic field is $3\,dx$ above the $x-y$ plane, and $2\,dz$ below. The surface current on the boundary is

$$
\begin{aligned}
J_s &= (3\,dx - 2\,dz)|_{z=0} \\
&= 3\,dx
\end{aligned}
$$

The current flows in the y direction along the lines of dx in the $x-y$ plane.

Example 3.10. Surface current on a paraboloid

If a magnetic field $H = dz$ exists above the paraboloid $z = x^2 + y^2$, and the field is zero below, the magnetic field boundary condition requires that a surface current

$$
\begin{aligned}
J_s &= dz|_{z=x^2+y^2} \\
&= d(x^2 + y^2) \\
&= 2\,dx + 2\,dy
\end{aligned}
$$

flow on the paraboloid.

In a similar manner, we can show that the electric field satisfies the boundary condition

$$
(E_1 - E_2)|_B = 0 \tag{3.26}
$$

This condition requires that the tangential component of the electric field above and below a boundary must be equal at the boundary.

Example 3.11. Electric field boundary condition

Is it possible for the magnetic field to be equal to dx for $x > \cos y$ and $-dx$ for $x < \cos y$? We apply the electric field intensity boundary condition,

$$
\begin{aligned}
(E_1 - E_2)|_{x=\cos y} &= 2d\cos y \\
&= -2\sin y\,dy \\
&\neq 0
\end{aligned}
$$

so that the boundary condition is not satisfied, and this field configuration cannot exist.

3.6.2 Flux Density

From Gauss's law, it can be shown that the electric flux density satisfies the boundary condition

$$
(D_1 - D_2)|_B = \rho_s \tag{3.27}
$$

where ρ_s is a 2-form representing the density of electric surface charge on the boundary. This 2-form is represented graphically as boxes which are the intersection of the tubes of $D_1 - D_2$ with the boundary, as in Fig. 3.9. If the tubes of the magnetic flux discontinuity are parallel to the boundary, then the tubes do not intersect and the restriction is zero. The left–hand side of Eq. (3.27) is the component of the jump in flux which is normal to the boundary.

The magnetic flux density satisfies the boundary condition

$$
(B_1 - B_2)|_B = 0 \tag{3.28}
$$

so that the normal components of the magnetic flux above and below a boundary must be equal.

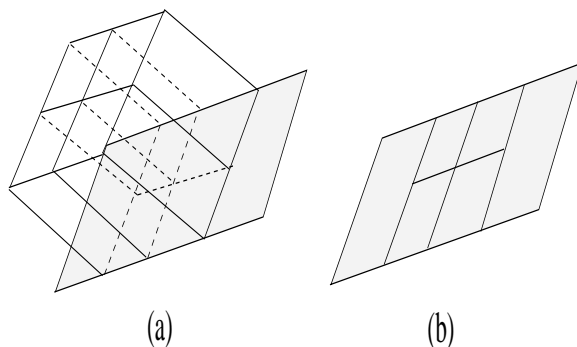**Example 3.12.** Electric Flux Density

Figure 3.9: (a) The 2-form $D_1 - D_2$. (b) The 2-form ρ_s, represented by boxes on the boundary.

An electric field of $dz\, dx$ exists for $z > x^2 + y^2$, and below the paraboloid the field is zero. The surface charge on the paraboloid is

$$
\begin{aligned}
\rho_s &= dz\, dx|_{z=x^2+y^2} \\
&= d(x^2 + y^2) \wedge dx \\
&= -2y\, dx\, dy
\end{aligned}
$$

For $y < 0$, the tubes of electric flux are oriented away from the boundary and the surface charge is positive, and for $y > 0$, the tubes point towards the boundary, and the surface charge is negative.

We collect all of the boundary conditions for reference:

$$
\begin{aligned}
(E_1 - E_2)|_B &= 0 \\
(H_1 - H_2)|_B &= J_s \\
(D_1 - D_2)|_B &= \rho_s \\
(B_1 - B_2)|_B &= 0
\end{aligned}
$$

The first two involve the tangential component of the field intensity, and the second pair involve the normal component of flux density. These conditions are simply convenient restatements of Maxwell's laws for fields at a boundary.

3.7 Exercises

3.1. Let $a = x + yz$, $b = x^2 \, dx$ and $b = yz \, dz \wedge dx$. (a) Find the exterior derivatives da, db and dc.

3.2. Find the exterior derivative of $\frac{yz}{\sqrt{x^2 y}} \, dx \wedge dy \wedge dz$.

3.3. Find the exterior derivatives of (a) $x - z^2$, (b) $x \, dx + y \, dy + z \, dz$ and (c) $x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$.

3.4. Find the exterior derivatives in spherical coordinates of the forms (a) $f \, d\theta$, (b) $r^2 \, dr + \sin \phi \, d\theta$, (c) $r \cos \theta \, d\theta \, d\phi$.

3.5. Find the exterior derivatives in cylindrical coordinates of the forms (a) $f \, d\rho$, (b) $\rho^2 \, d\phi + \sin \phi \, dz$, (c) $\rho z \, d\phi \, dz + z\rho \cos \phi \, d\rho \, d\phi$.

3.6. Derive Ampere's law in integral form from its point form.

3.7. (a) Integrate $x \, dy$ in a counterclockwise direction over the path formed by the edges of a square with corners at $(0, 0)$, $(0, 2)$, $(2, 0)$, and $(2, 2)$. (b) Integrate the exterior derivative of $x \, dy$ over the interior of the same square.

3.8. Let the electric field intensity be $E = e^{j\omega t}(x \, dx + y \, dy)$. Find the magnetic flux density B.

3.9. Derive the boundary condition for the electric flux density from Gauss's law for the electric field.

3.10. If the magnetic field intensity is $4 \, dx$ for $z > x$ and $-6 \, dx$ for $z < x$, find the surface current on the boundary $z = x$.

3.11. If the electric flux density is $dx \, dy$ for $z > \sin x$ and $-3 \, dx$ for $z < \sin x$, find the charge density on the boundary $z = \sin x$.