Discrete Exterior Calculus

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Introduction

Introduction

Motivation

- Exterior calculus is the language of modern mechanics. With *discrete mechanics* we need a *discrete exterior calculus (DEC).*
- If a problem solution satisfies *structural constraints* like being divergence free, then DEC is likely to be useful. Especially for *non-regular grids.*
- Many examples in mimetic differencing literature, Douglas Arnold's example of resonant frequencies in electromagnetic cavity etc. show importance of DEC-like methods. But they have no systematic DEC development.
- Our other work in template matching, discrete shells and vector field decomposition would benefit greatly from DEC.

Introduction

Overall goal of this thesis

Start laying the foundation of a discrete theory of exterior calculus.

Results of this thesis

- Previous works have included either vector fields or differential forms. In our theory *both* are included.
- For the first time in this field, we have introduced discrete sharps and flats – operators connecting forms and vector fields.
- We show that the use of *circumcentric* dual meshes makes possible a theory with forms and vector fields.
- We introduce discrete interior product (contraction) and Lie derivatives – important in applications involving flows.
- We give some early results about discrete pullbacks.

Results (contd.)

- Formulas we derive for divergence, 2D curl and Laplace-Beltrami are identical to those found by other means.
- There is a discrete divergence theorem in DEC.
- In other work we have derived a PDE for image matching, modeled discrete shells and derived discrete vector field decomposition. There are clear signs that these will connect to DEC in the future.

Introduction

■ Limitations

- No convergence or stability studied !
- No numerical tests were done by us in the core DEC part.
- The space of all possible discrete sharps and flats has not been explored. In particular we haven't found an inverse pair.
- We found at least 2 definitions of the interior product and Lie derivative. We don't know if they are identitcal.
- Current DEC theory places restrictions on types of meshes : we don't know if these can be lifted.
- DEC as developed so far works only with nearest neighbor interactions and yields "lowest order" formulas. We don't know if this is a fundamental limitation.

Introduction

Discretization

- Approximate manifold M by a cell complex K and its dual cell complex.
- Define discrete forms, vector fields and operators on these complexes.
- DEC is *not* exterior calculus on charts. It is a geometric and combinatorial calculus.
- Difference from discrete mechanics : discretization spread.
- Unique in discrete theory : duals result in operator proliferation.

Background

Exterior Calculus Operators

Given a smooth manifold M the following spaces and operators are defined :

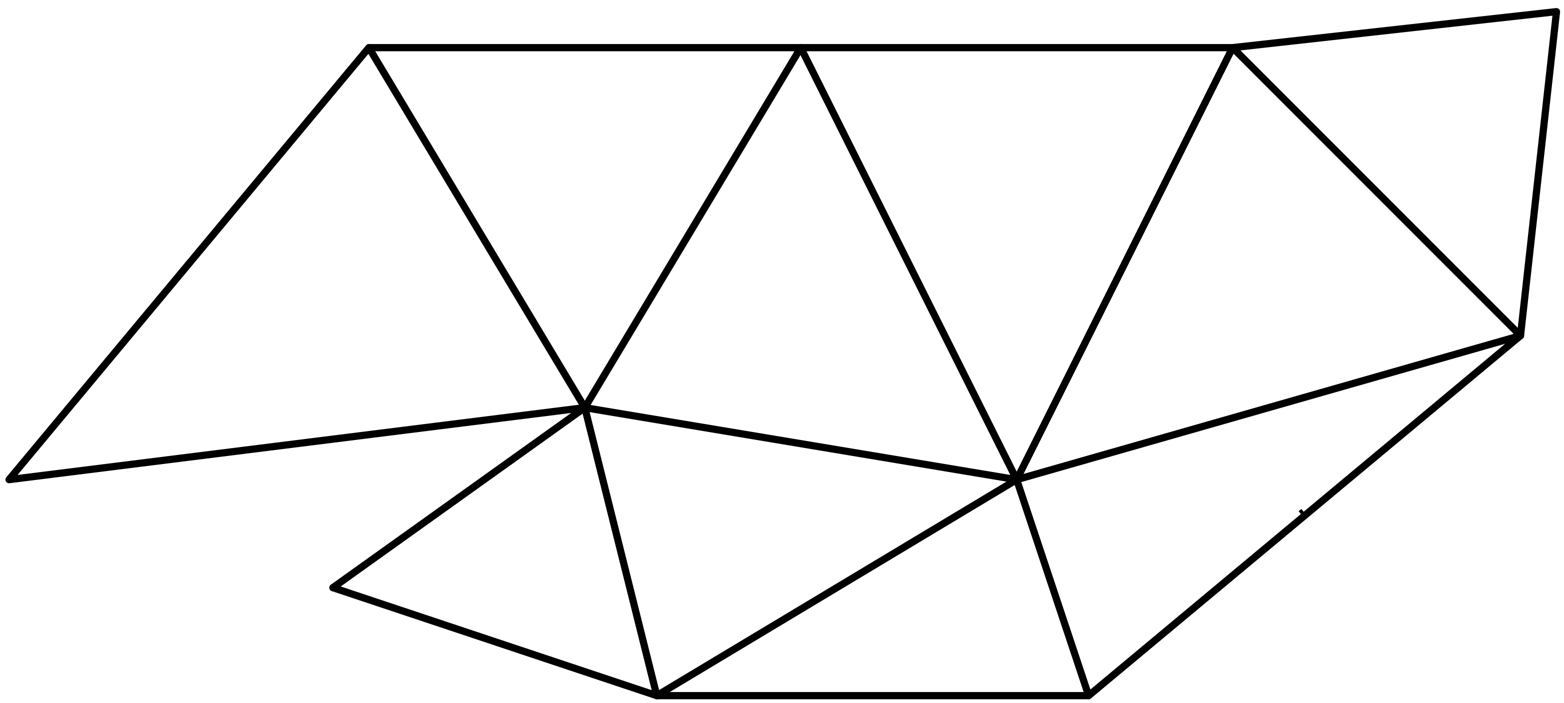
- p-forms in $\Omega^p(M)$ and vector fields in $\mathfrak{X}(M)$
- Exterior derivative $\mathbf{d} : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$
- Hodge star $* : \Omega^p(M) \longrightarrow \Omega^{n-p}(M)$
- Wedge product $\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$
- Sharp map $\sharp : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$
- Flat map $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$
- Interior product (contraction) i_X of forms with vector field
- Lie derivative $\pounds_X$ of forms and vector fields

History and Previous Work

- Physics : Tonti [2002]; Sen et al. [2000]; Schwalm et al. [1999]
- Computational Electromagnetism : Bossavit [2002, 2001]; Hiptmair [2001, 2002]; Teixeira [2001]; Gross and Kotiuga [2001]
- Mimetic Discretization : Hyman and Shashkov [1997a,b]
- Numerical Analysis : Arnold [2003]; Mattiussi [1997]
- Computer Graphics : Meyer et al. [2002]; Gu [2002]
- Mathematics : Dezin [1995]; Mansfield and Hydon [2001]; Harrison [1993]

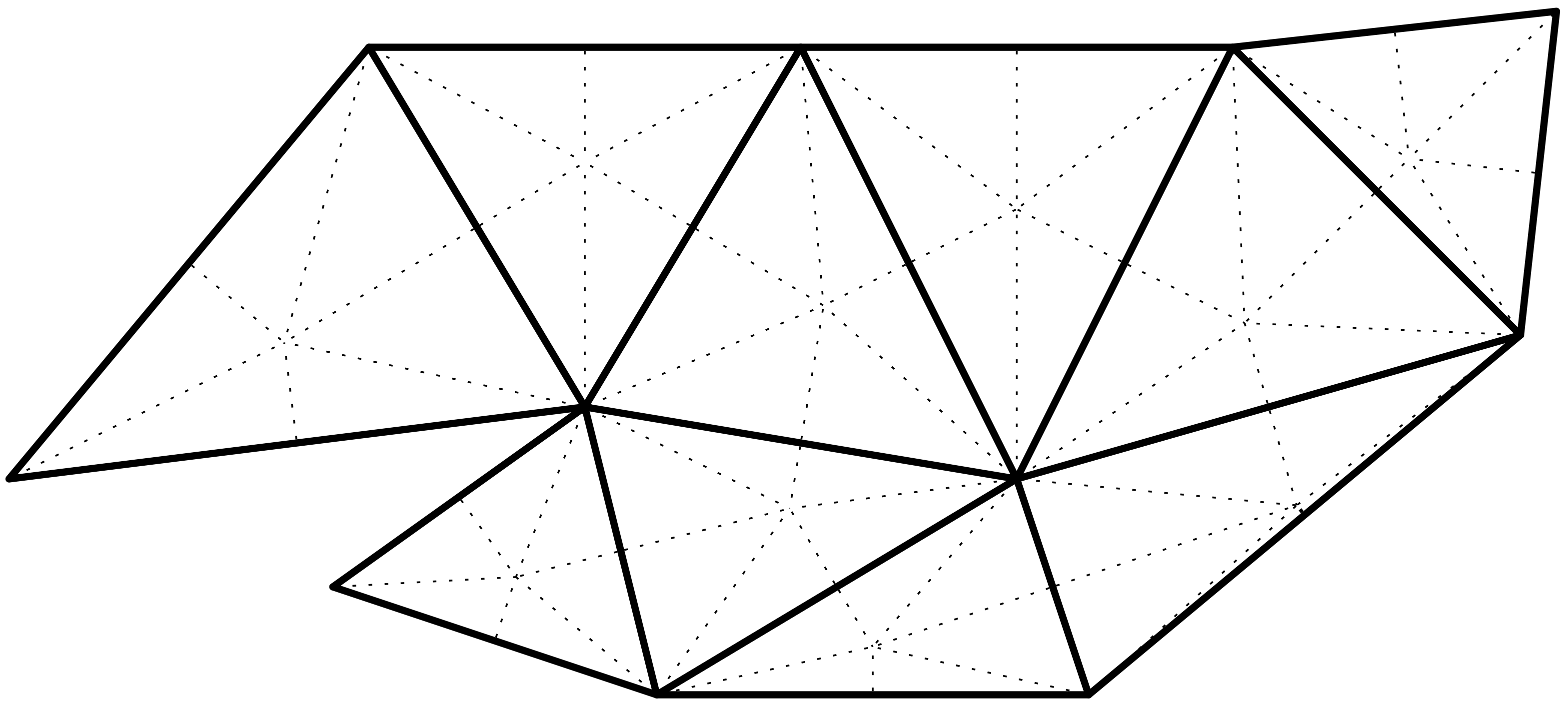
Primal and Dual Complexes

- **A simplicial complex (doesn't have to be flat)**



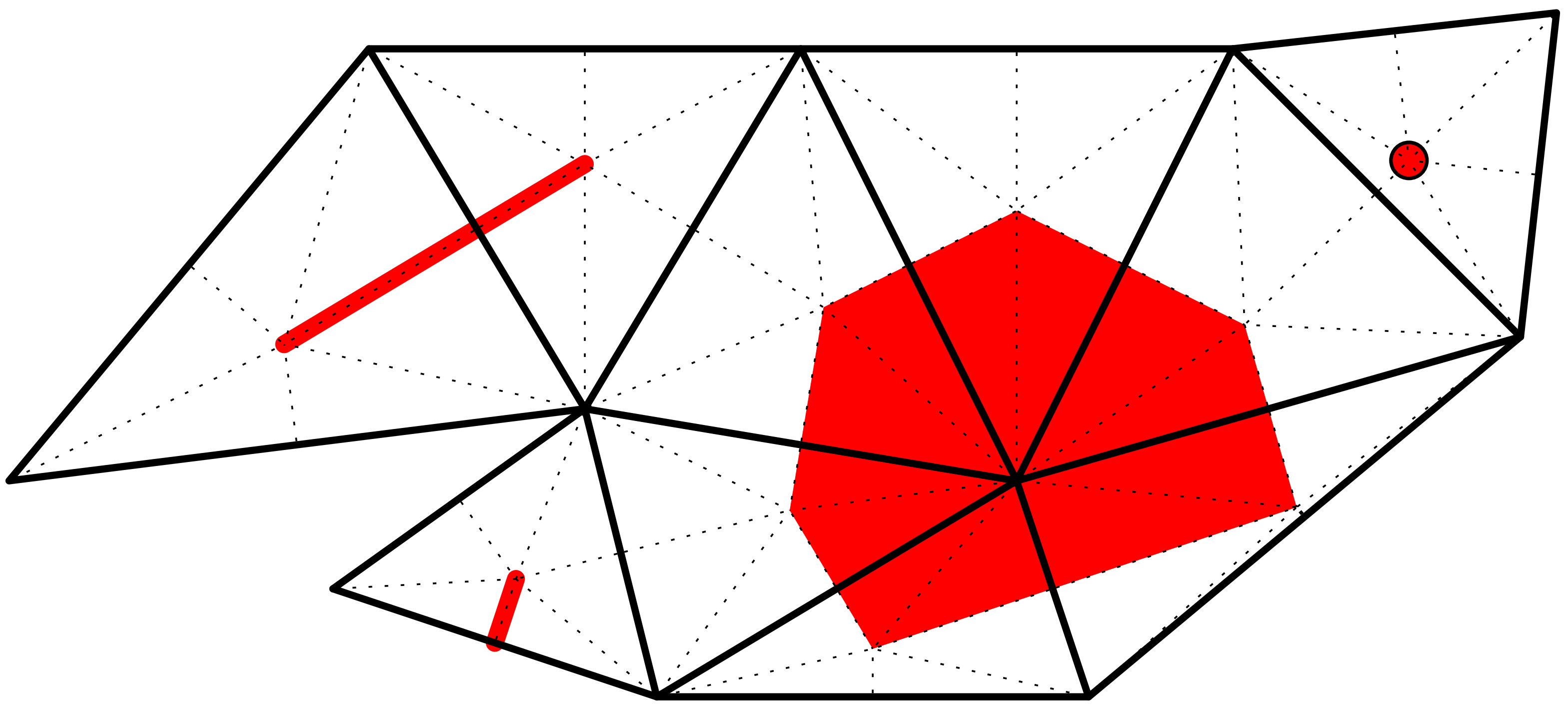
Primal and Dual Complexes

- **Circumcentric subdivision of the complex**



Primal and Dual Complexes

- **Examples of dual cells**



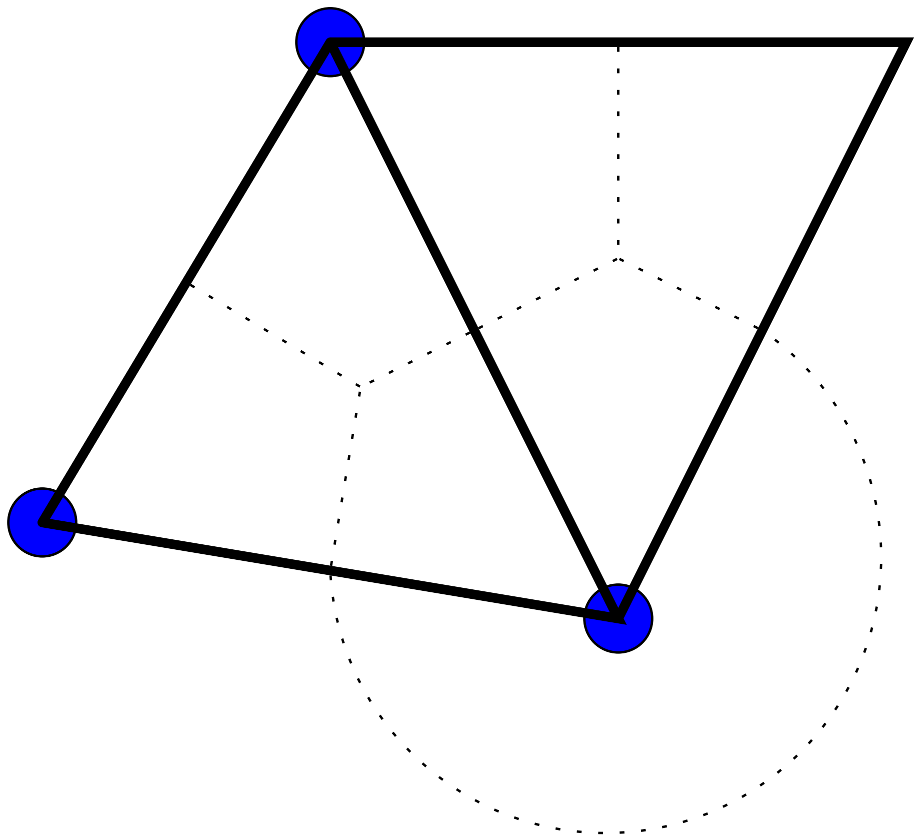
Primal and Dual Complexes

Restrictions and notations

- Restriction : circumcenters must lie inside each simplex ; complex must be a C^0 oriented manifold.
- Simplices written as σ^0, σ^1, σ^2 etc. (superscript is dimension).
- $|\sigma^p|$ is p-volume, $D(\sigma^p)$ (dimension $n-p$) is dual of σ^p, $\star\sigma^p$ is dual with subdivision information attached. $\star\star\sigma^p := \pm\sigma^p$.
- $\sigma^q \prec \sigma^p$ means σ^q is a proper face of σ^p.
- K simplicial complex, $D(K)$ is dual, $\star K$ is dual with subdivision information, $|K|$ is underlying space with subspace topology.
- We found an easy computational interpretation of dual's orientation (in algebraic topology you need relative homology groups etc.).

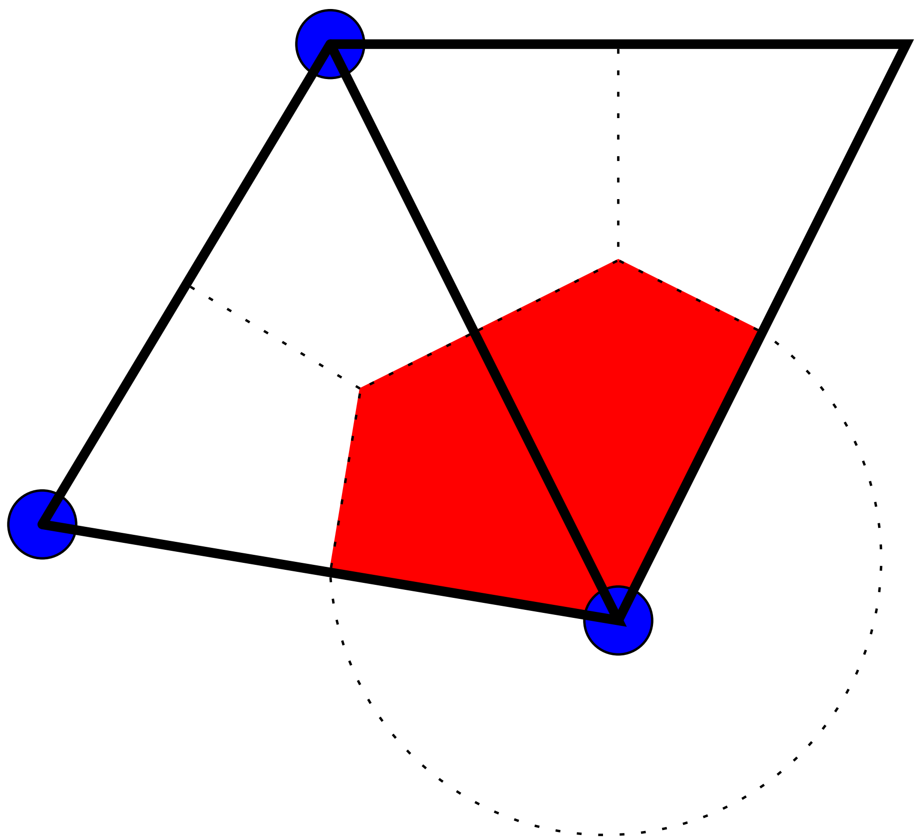
Four Types of Interpolation

- **Primal data**



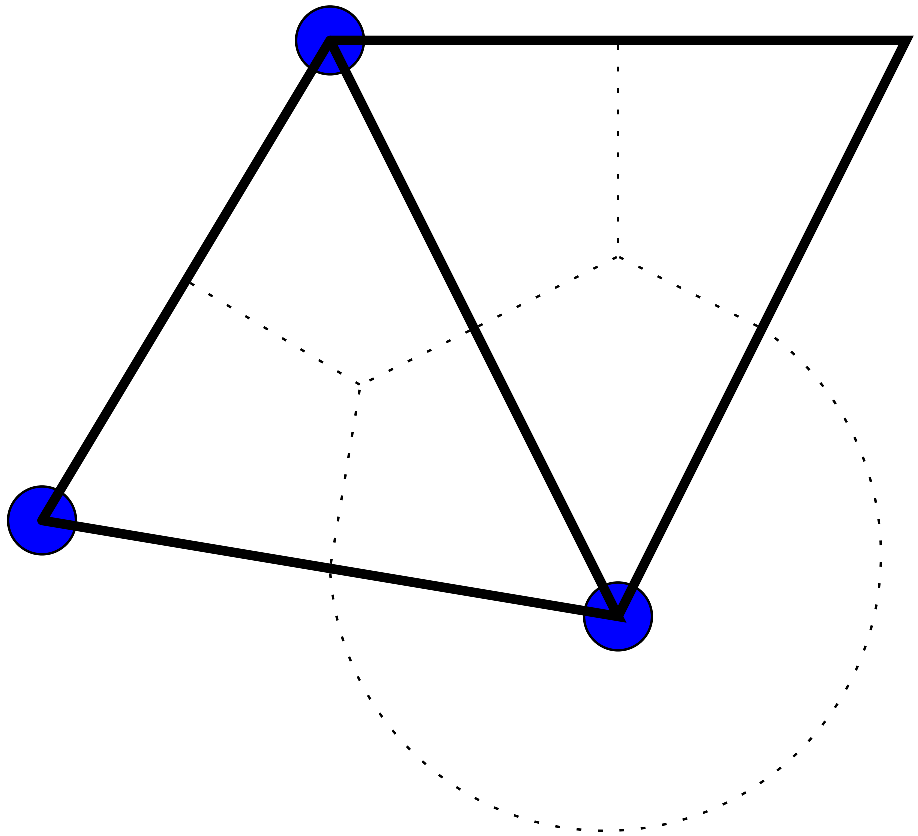
Four Types of Interpolation

- **Primal to dual interpolation : constant**



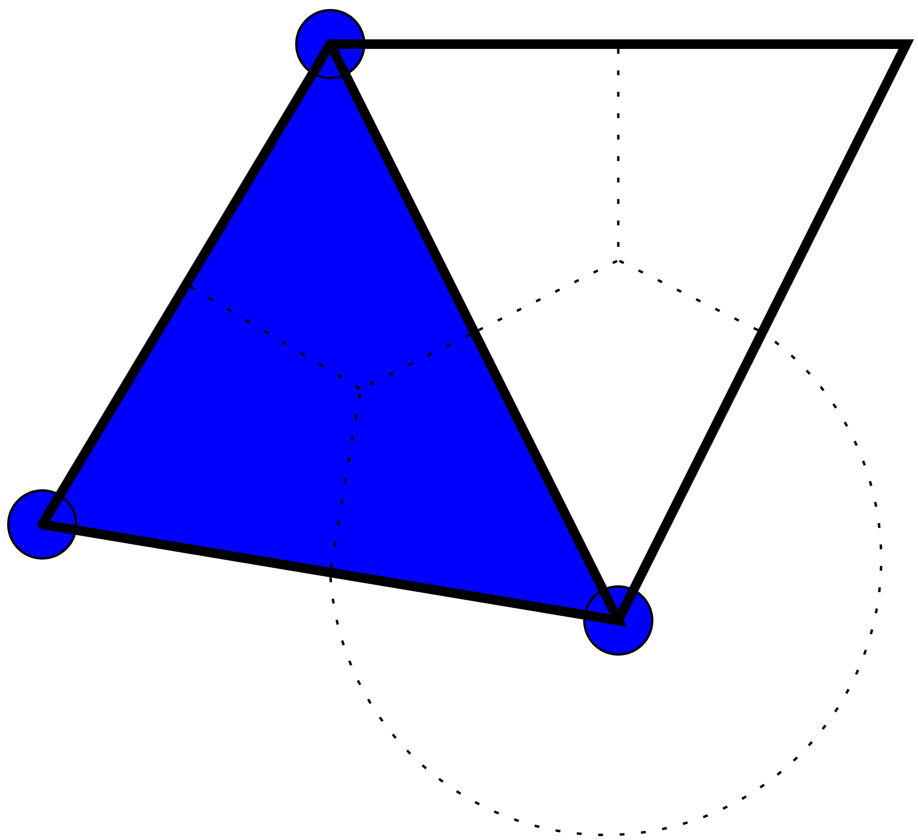
Four Types of Interpolation

- **Primal data**



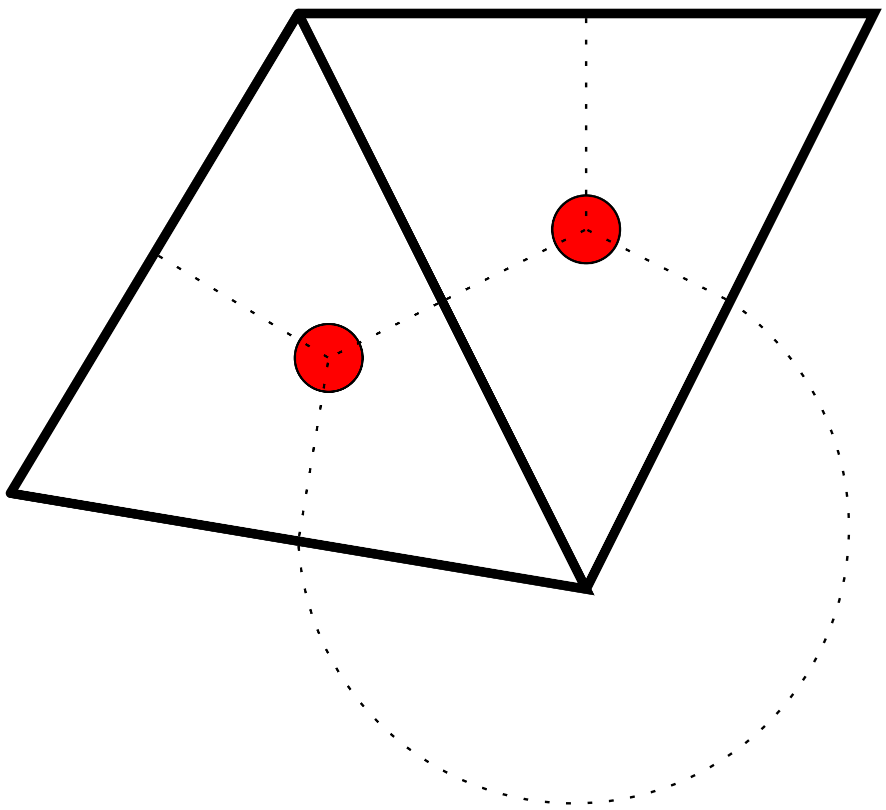
Four Types of Interpolation

- **Primal to primal interpolation : linear**



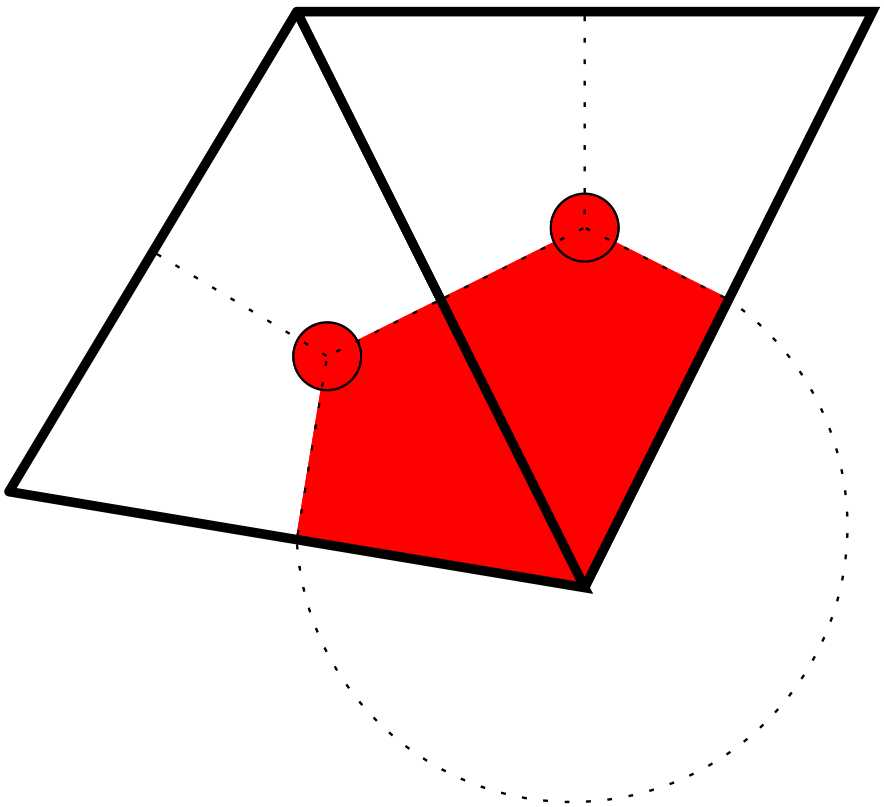
Four Types of Interpolation

- **Dual data**



Four Types of Interpolation

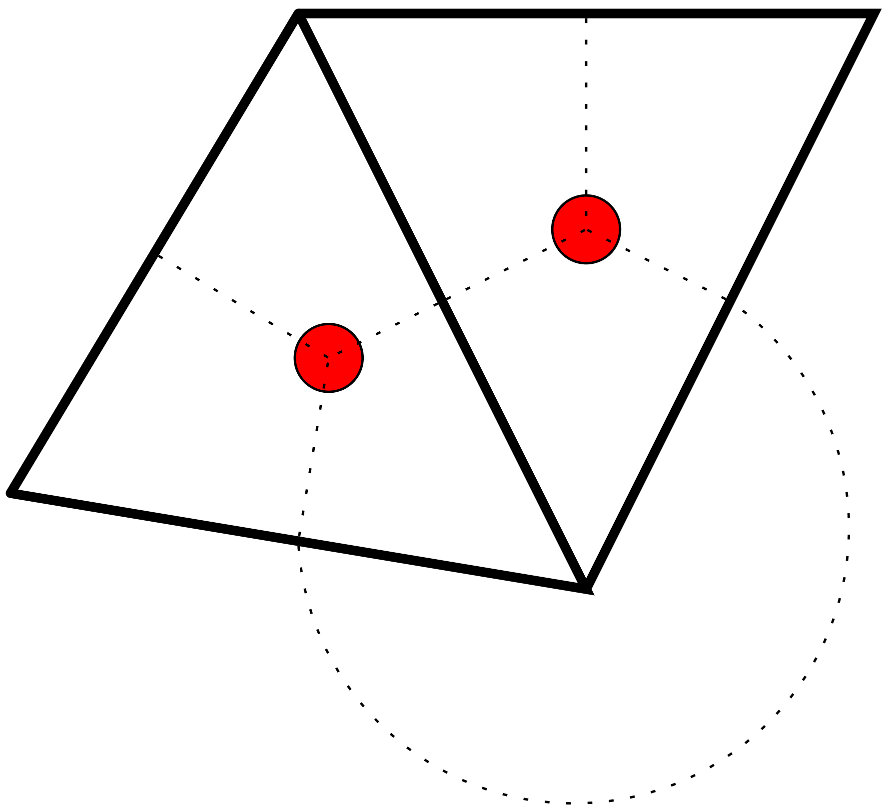
■ Dual to dual interpolation : nonlinear



- Based on our new barycentric coordinates for convex polyhedra Warren et al. [2003]. Mesh is assumed to be flat.

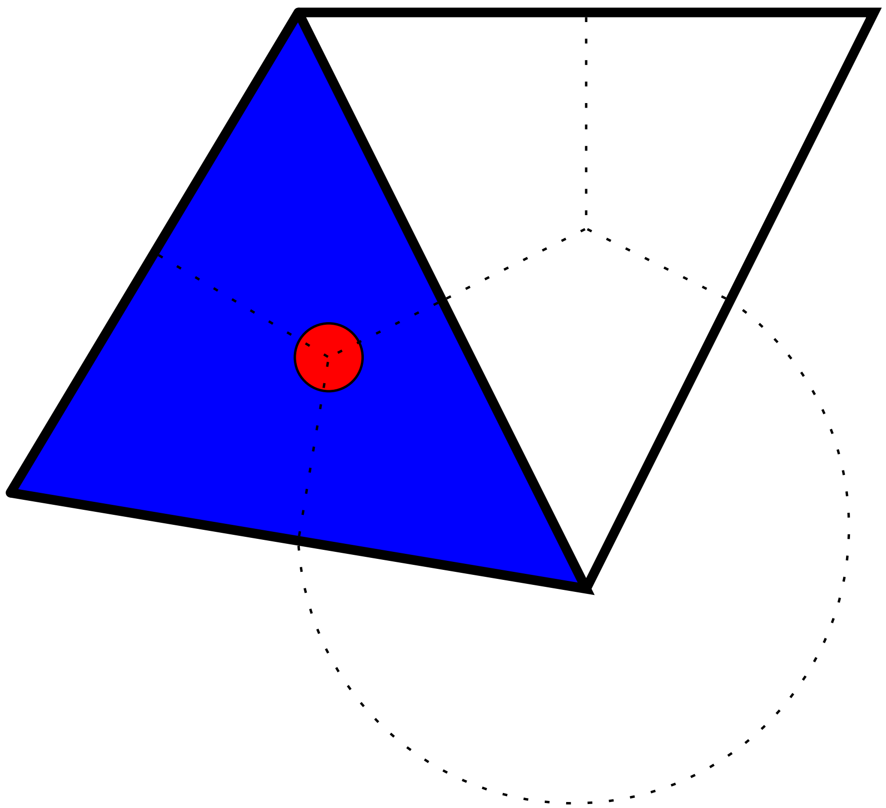
Four Types of Interpolation

- **Dual data**



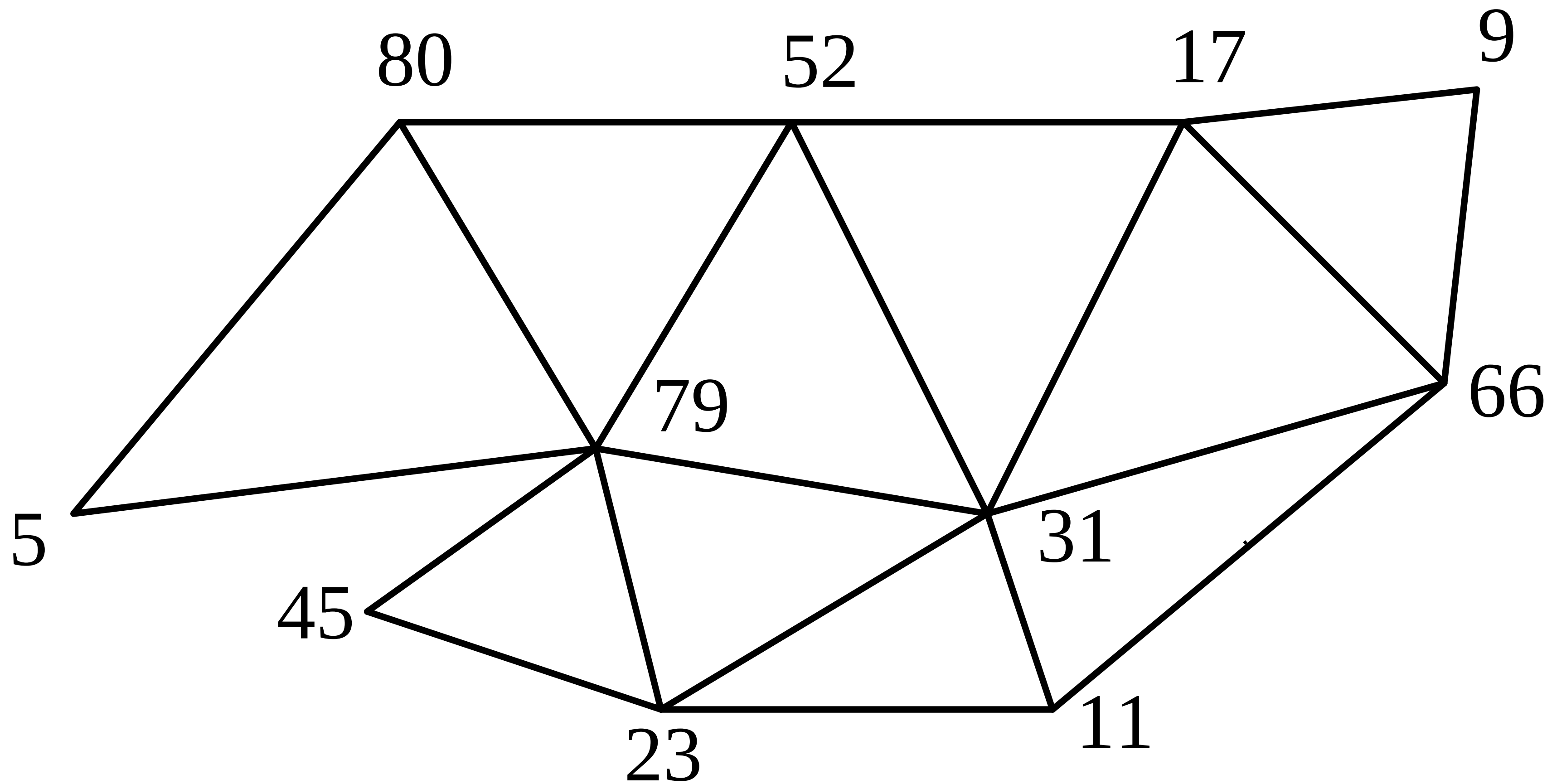
Four Types of Interpolation

- **Dual to primal interpolation : constant**



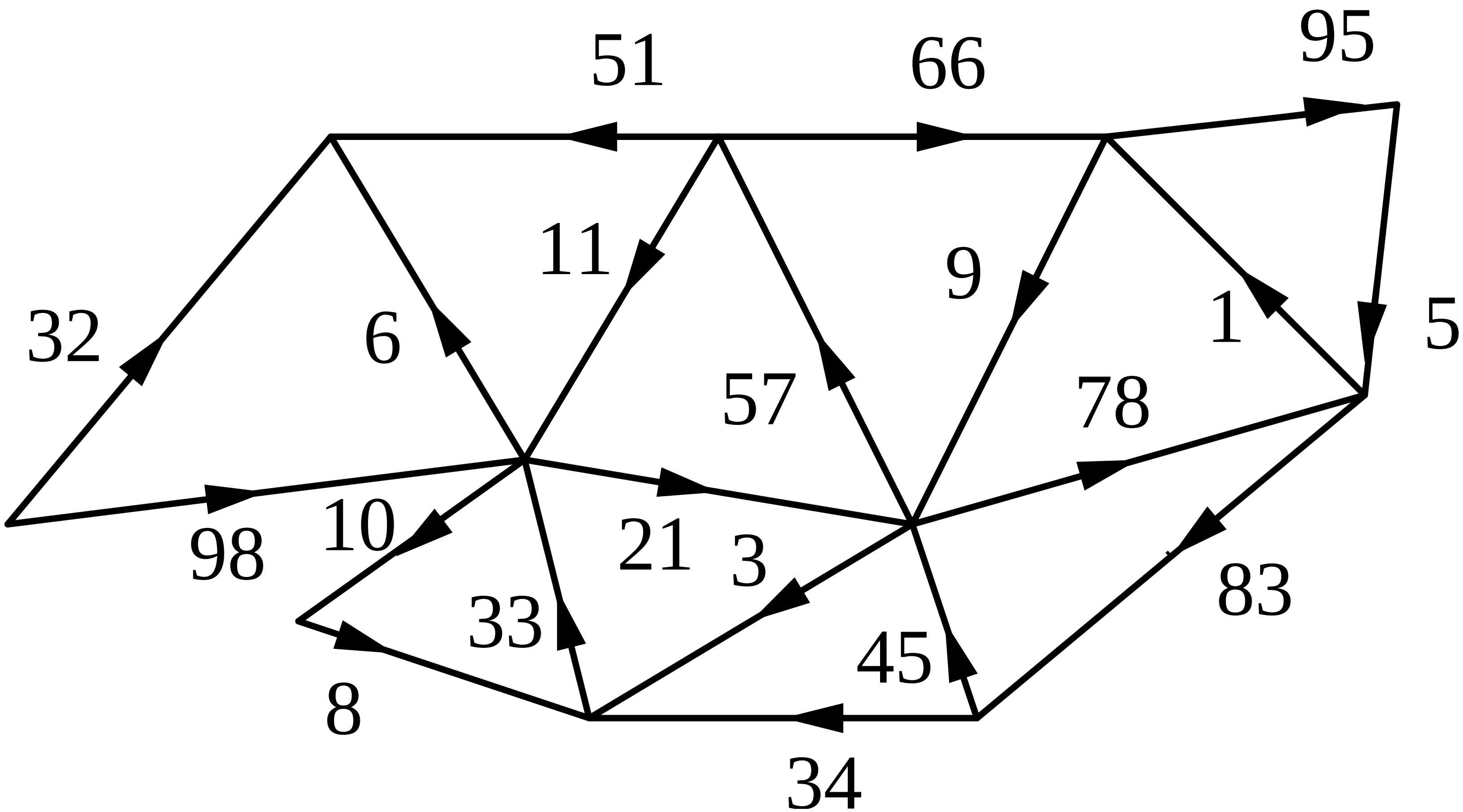
Discrete Forms

- **Example of a discrete 0-form : numbers on vertices**



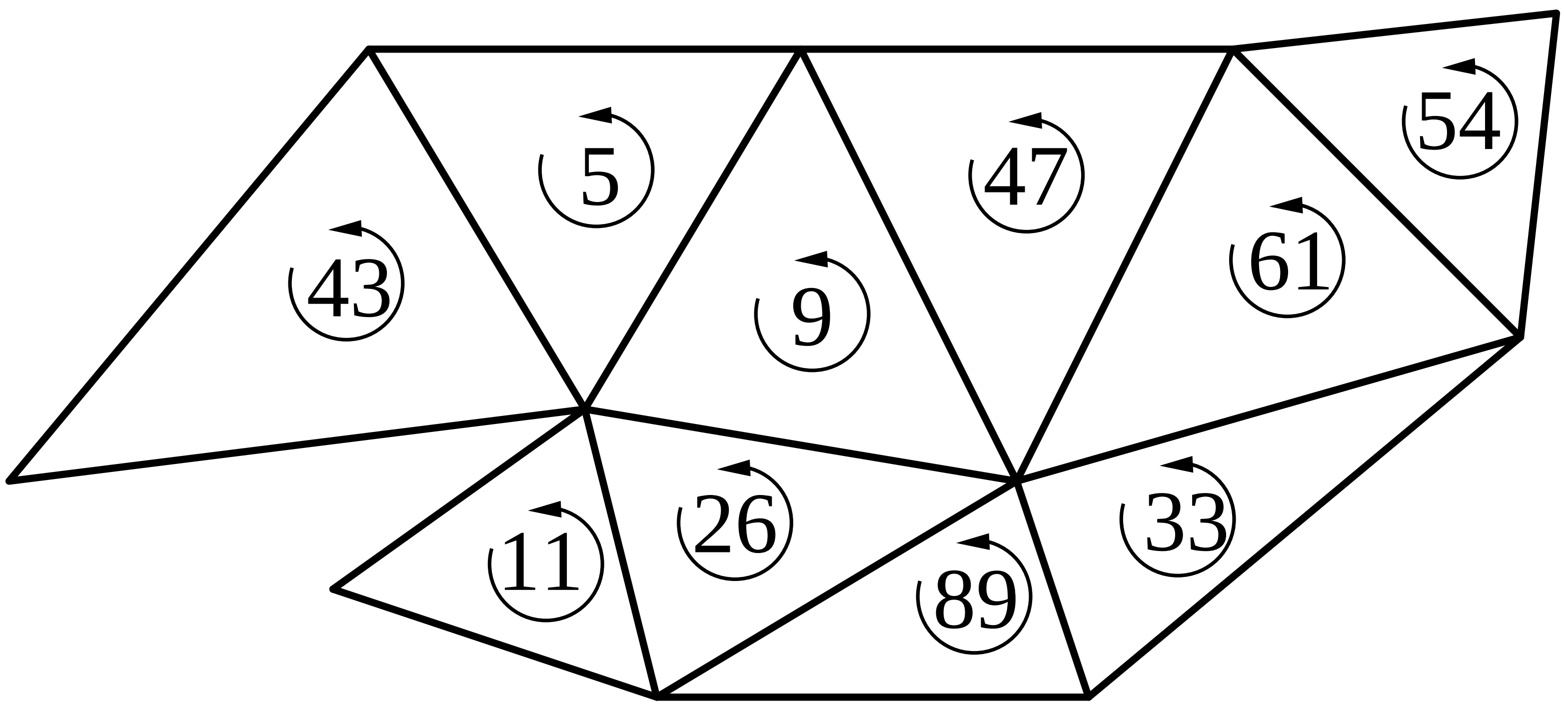
Discrete Forms

- **Example 1-form : numbers on oriented edges**



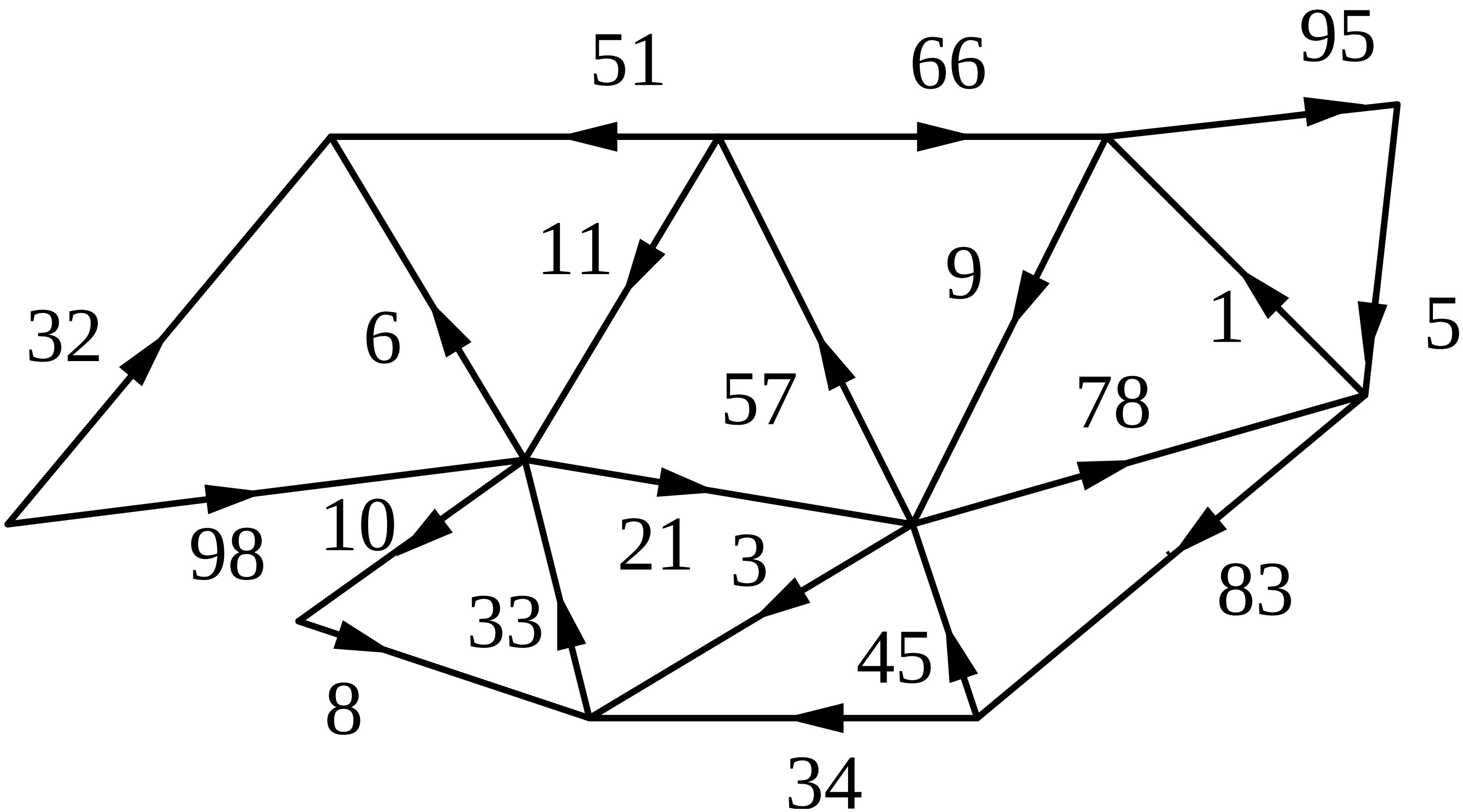
Discrete Forms

- **Example 2-form : numbers on oriented triangles**



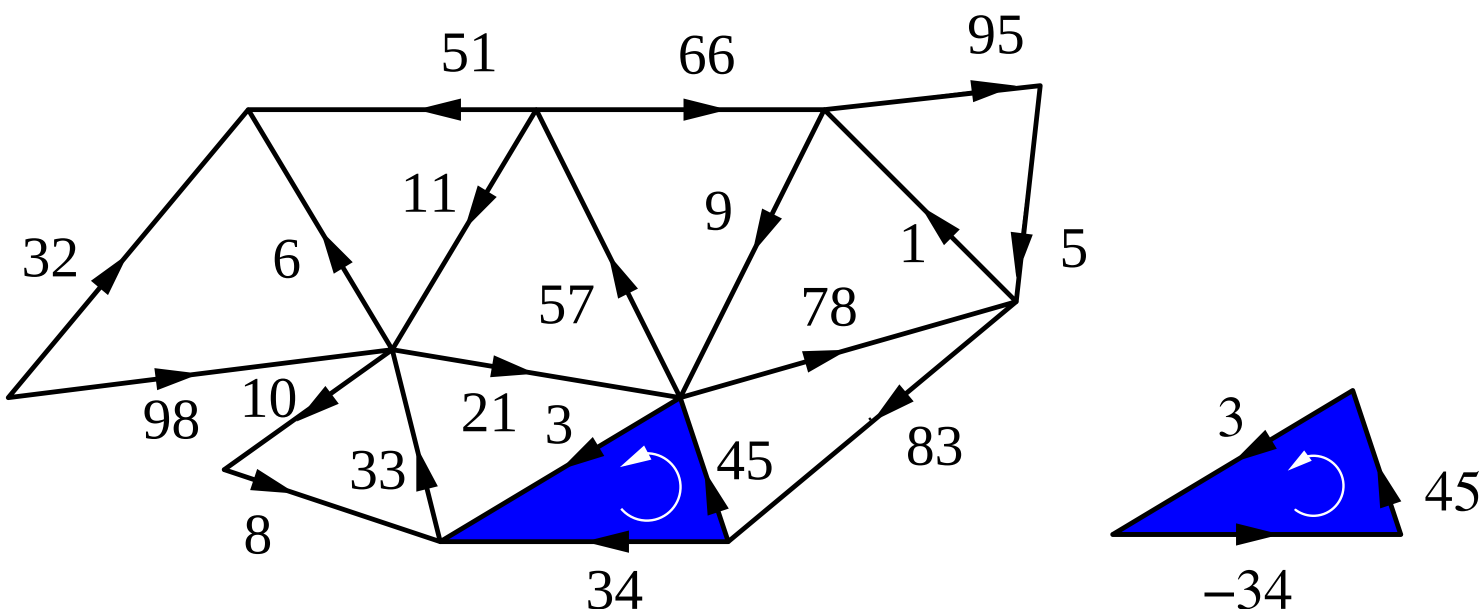
Exterior Derivative

- **Compute exterior derivative for this 1-form**



Exterior Derivative

■ Computed on blue triangle



- Just add up values shown on little triangle on right.
- Discrete **d** is the coboundary : $\langle \mathbf{d}\alpha^p, \sigma^{p+1}\rangle = \langle \alpha^p, \partial\sigma^{p+1}\rangle$
- Discrete Stokes' theorem is true by definition.
- $\mathbf{d} \circ \mathbf{d} = 0$.

Review of smooth flat and sharp

- Smooth sharp ($\sharp$) and flat are inverses of each other :

$$\alpha^\sharp \cdot V = \alpha(V)$$

$$X^\flat(V) = X \cdot V$$

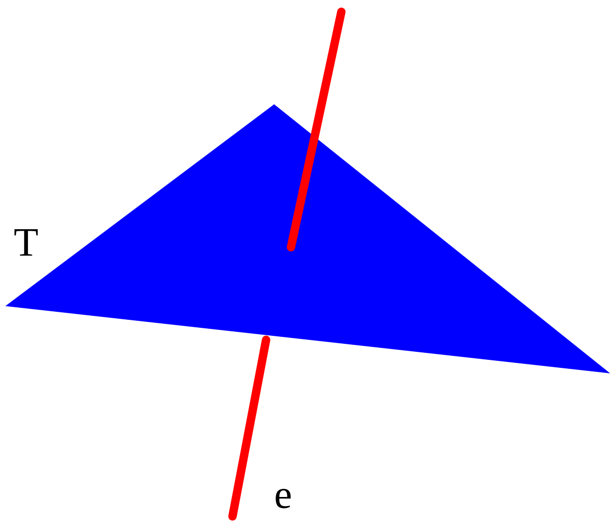
$$\left(X^\flat\right)^\sharp \cdot V = X^\flat(V) = X \cdot V$$

$$\left(\alpha^\sharp\right)^\flat (V) = \alpha^\sharp \cdot V = \alpha(V)\,.$$

- Gradient $\nabla f = (\mathbf{d}f)^\sharp$ or equivalently $(\nabla f)^\flat = \mathbf{d}f$.

Contributions

Hodge Star and Codifferential



$$\frac{1}{\text{Area}(T)}\langle \alpha, T\rangle = \frac{1}{\text{Length}(e)}\langle *\alpha, e\rangle$$

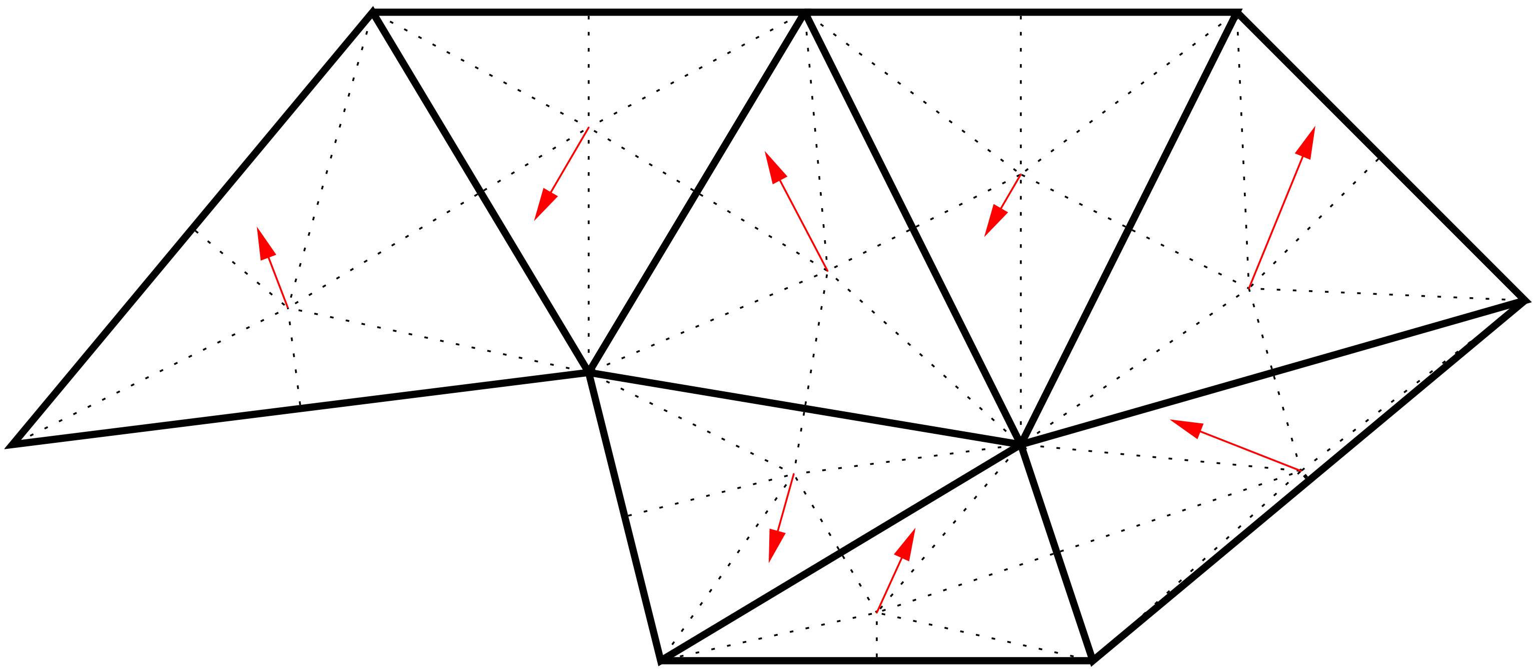
$$\frac{1}{|\sigma^p|}\langle \alpha, \sigma^p\rangle = \frac{1}{|\star\,\sigma^p|}\langle *\alpha, \star\sigma^p\rangle\,.$$

Hodge Star and Codifferential

- Information transfer between primal and dual.
- Used below to define codifferential.
- Codifferential used to define div, curl, Laplace-Beltrami.
- Codifferential $\boldsymbol{\delta}\beta = (-1)^{np+1} * \mathbf{d} * \beta$.

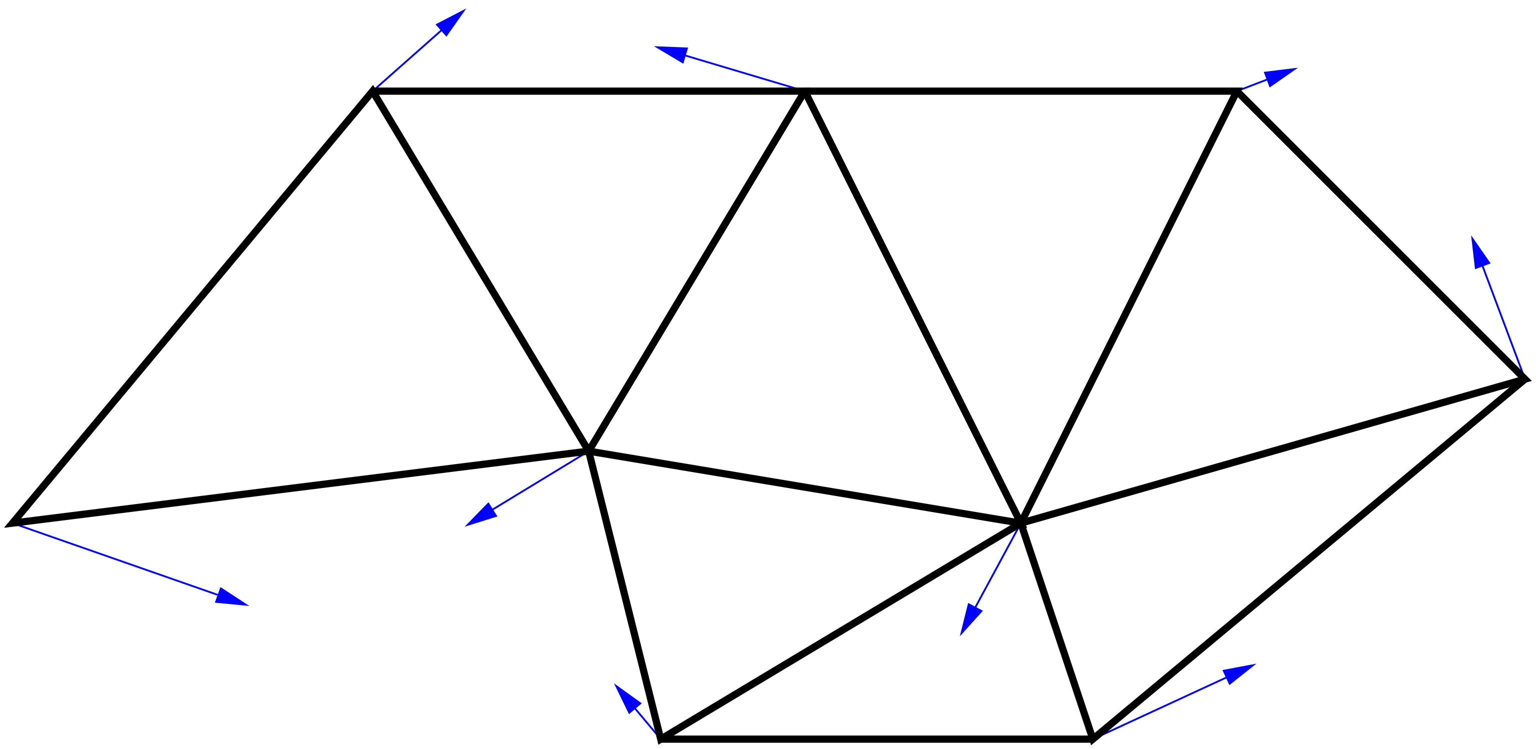
Discrete Vector Fields

- **A dual vector field : mesh can be non-flat**



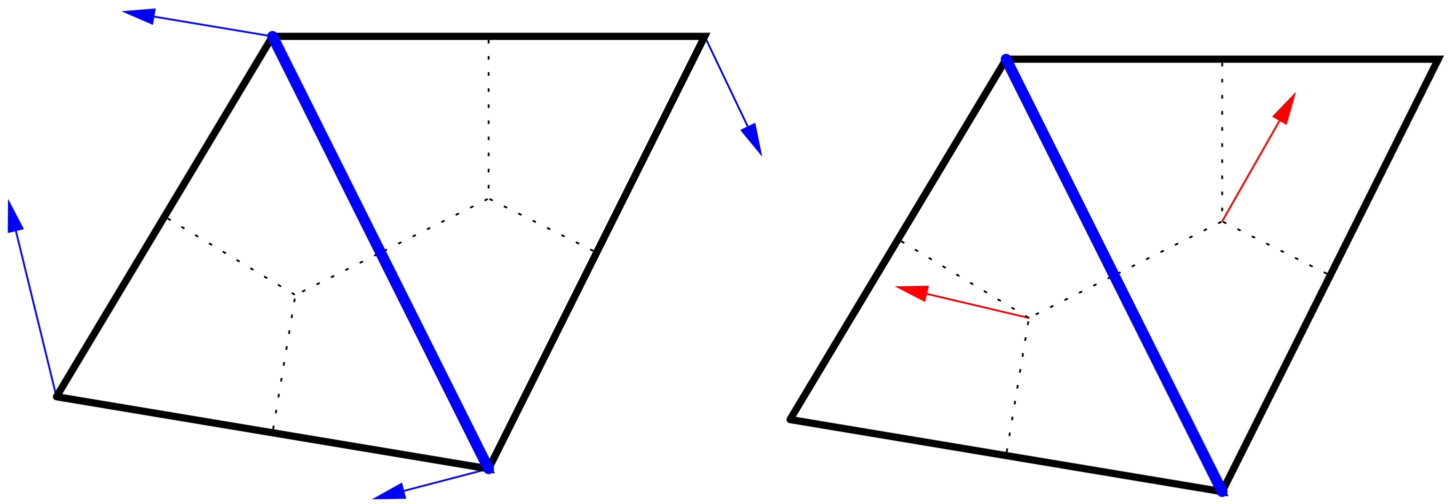
Discrete Vector Fields

- **A primal vector field : mesh must be flat**



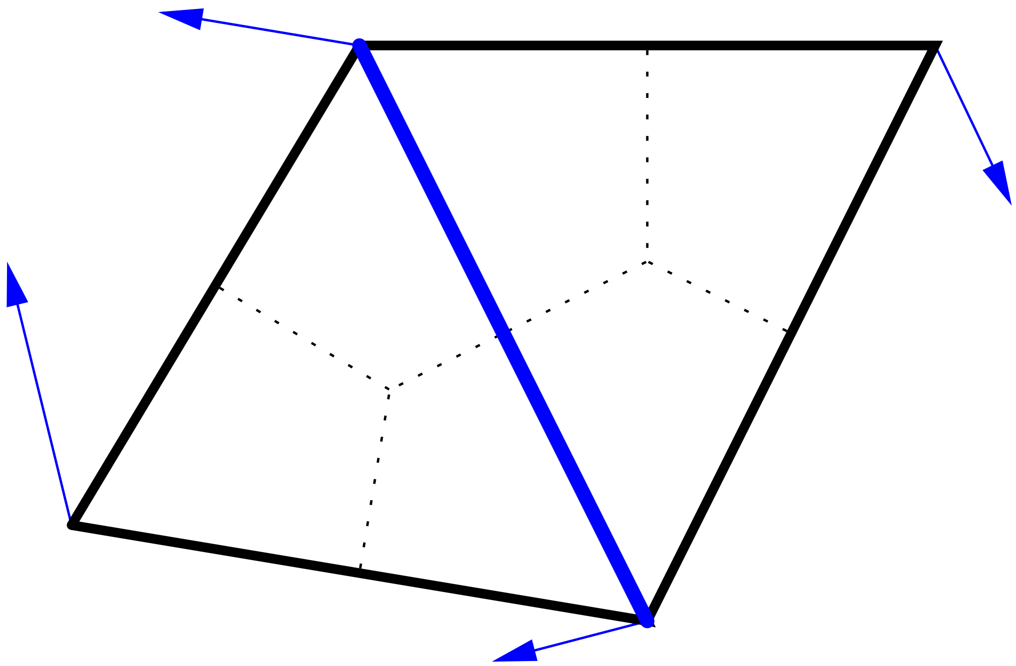
Flat Operator Connecting Forms and Vector Fields

■ Proliferation of Discrete Flats



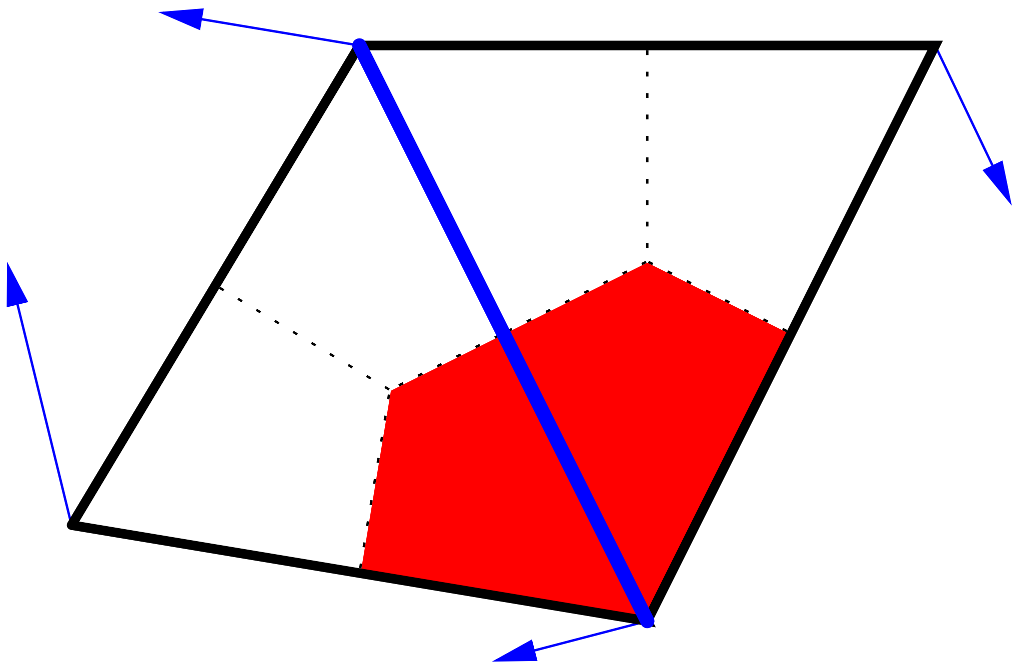
PDP Flat

- **Primal vector field $\longrightarrow$ Primal edges**



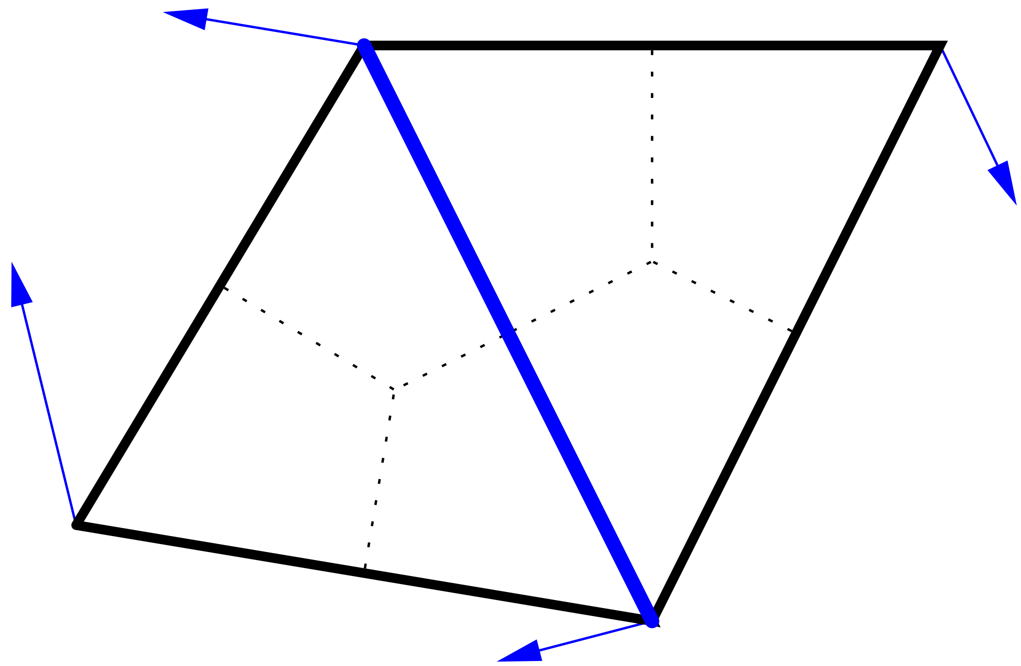
PDP Flat

- **Primal data ; Dual interpolation ; Primal destination**



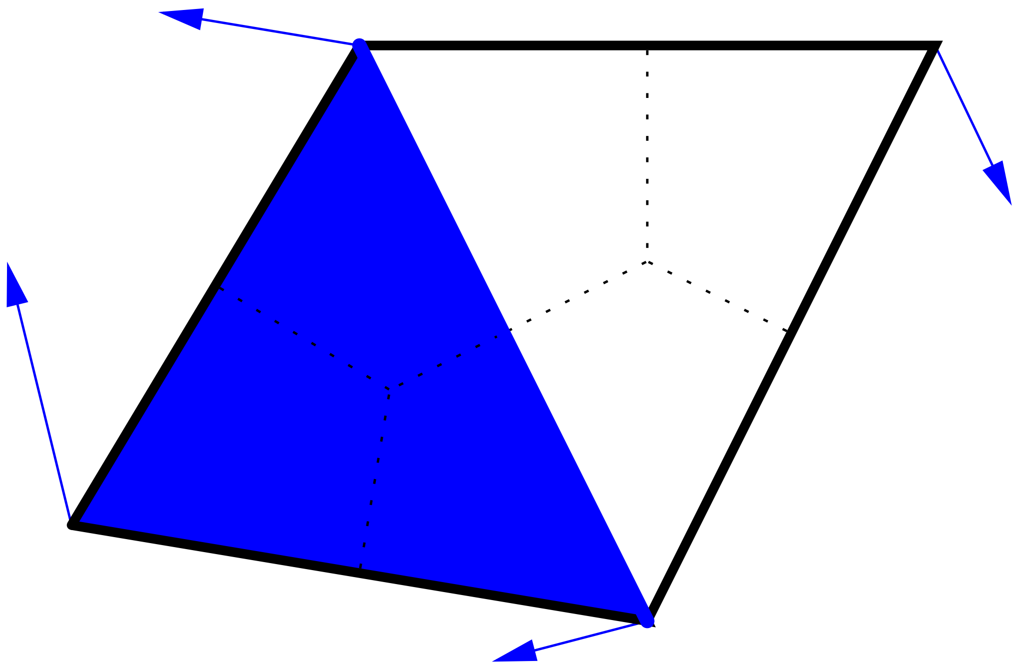
PPP Flat

- **Primal vector field $\longrightarrow$ primal edges**



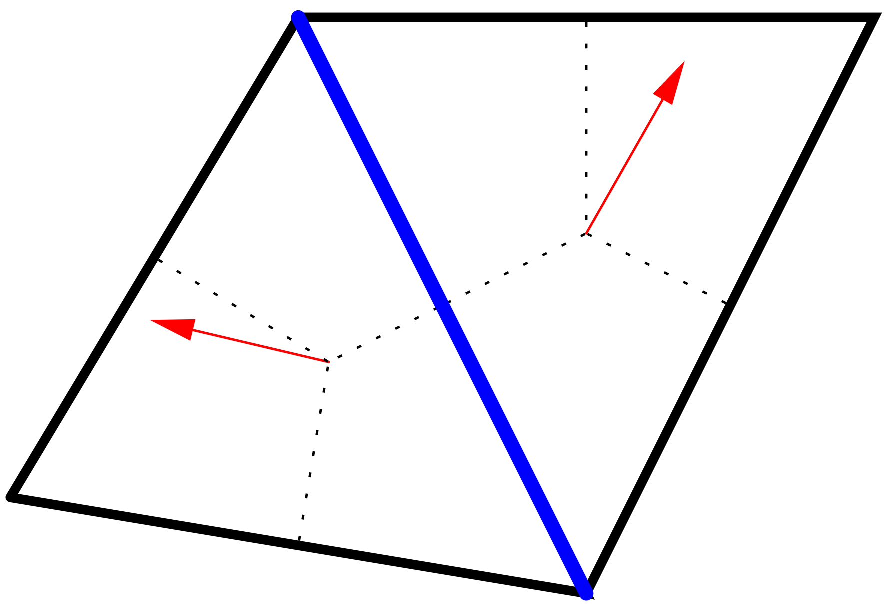
PPP Flat

- **Primal data ; Primal interpolation ; Primal destination**



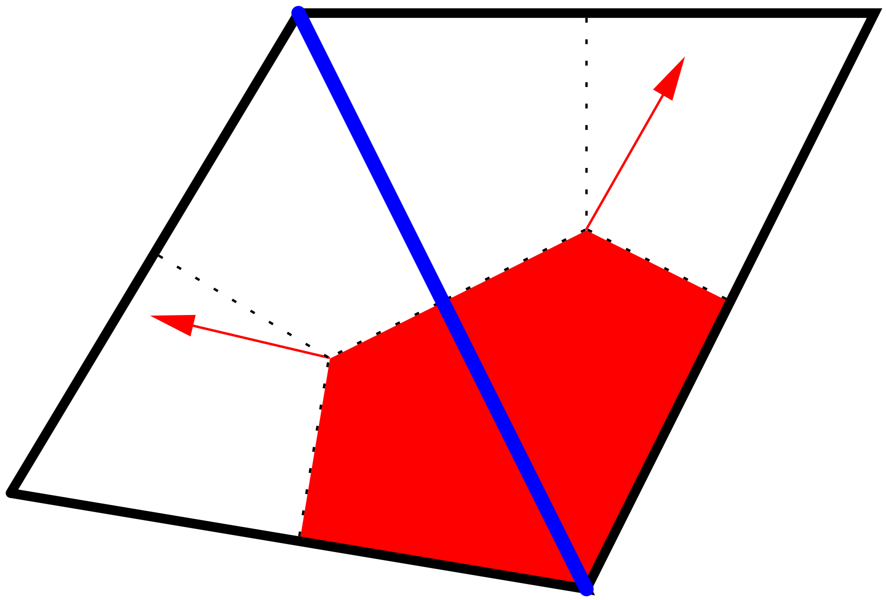
DDP Flat

- **Dual vector field $\longrightarrow$ Primal edges**



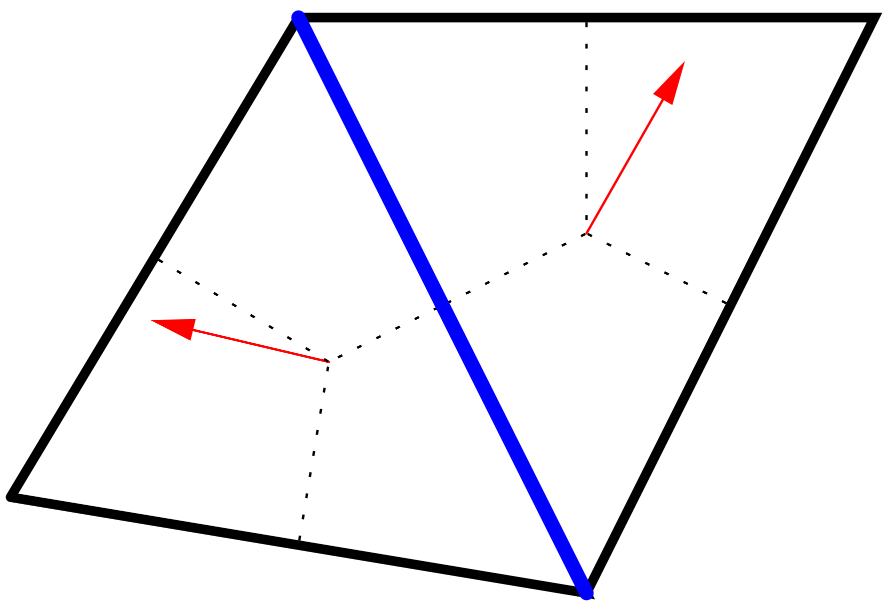
DDP Flat

- **Dual data ; Dual interpolation ; Primal destination**



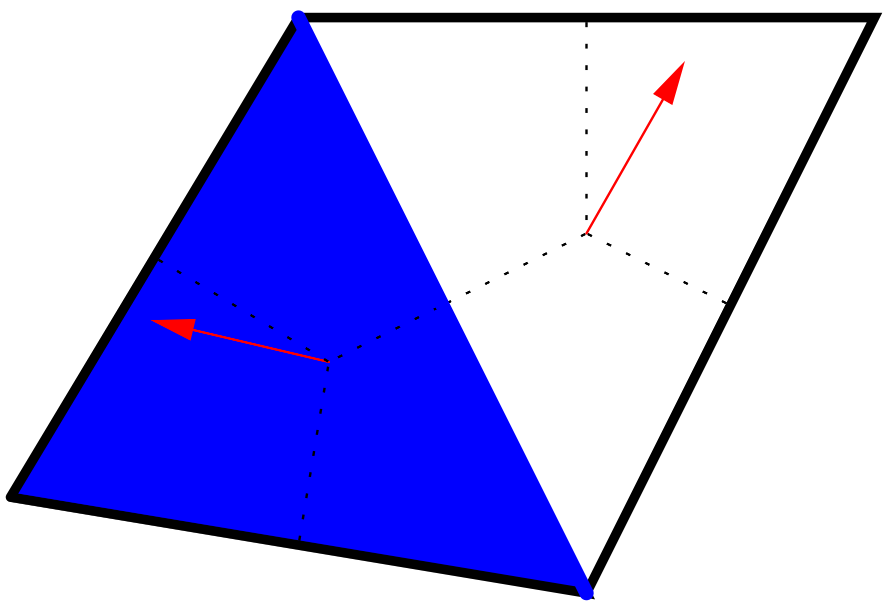
DPP Flat

- **Dual vector field $\longrightarrow$ Primal edges**

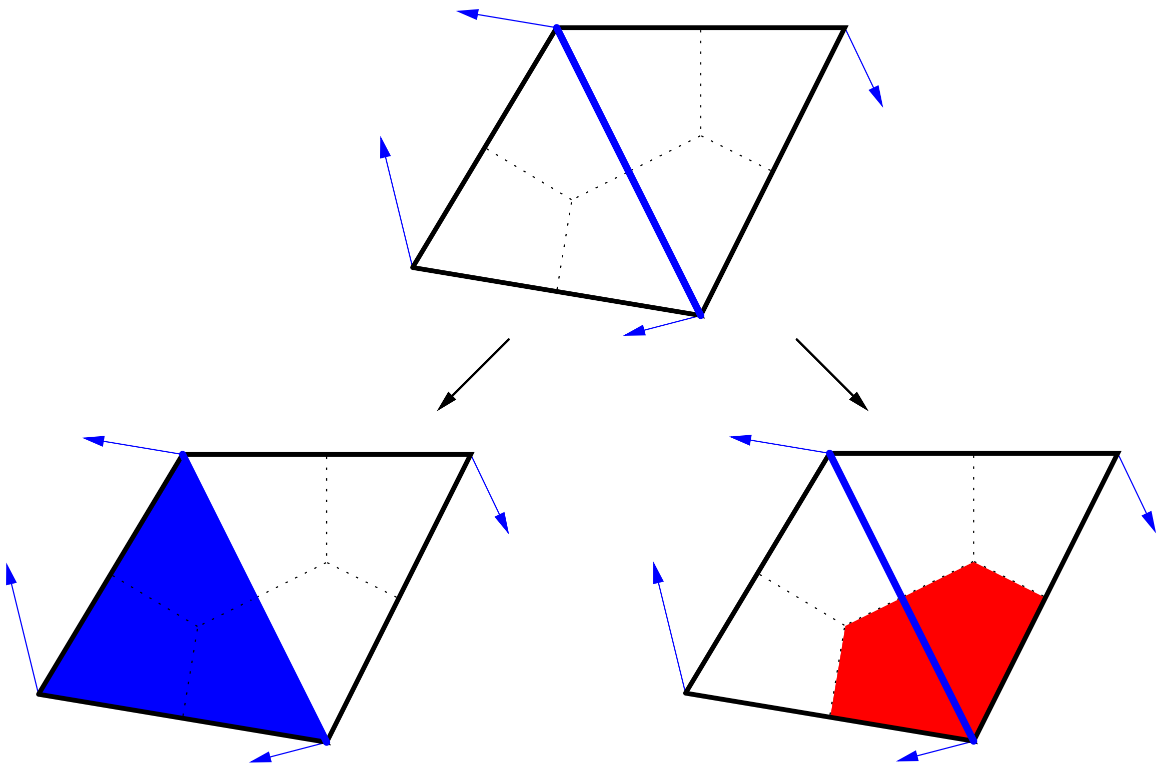


DPP Flat

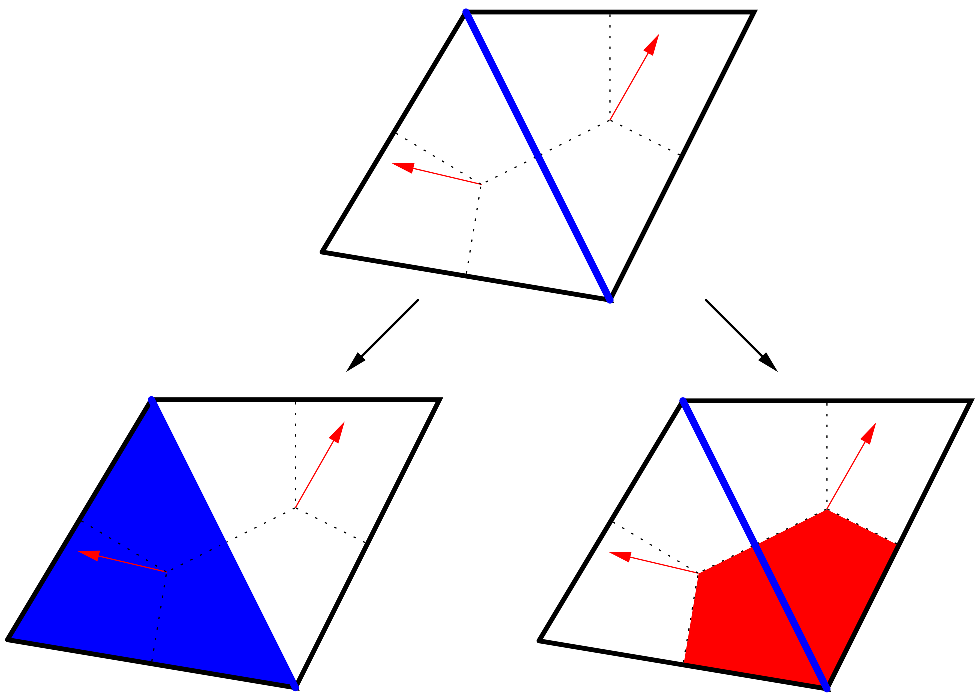
- **Dual data ; Primal interpolation ; Primal destination**



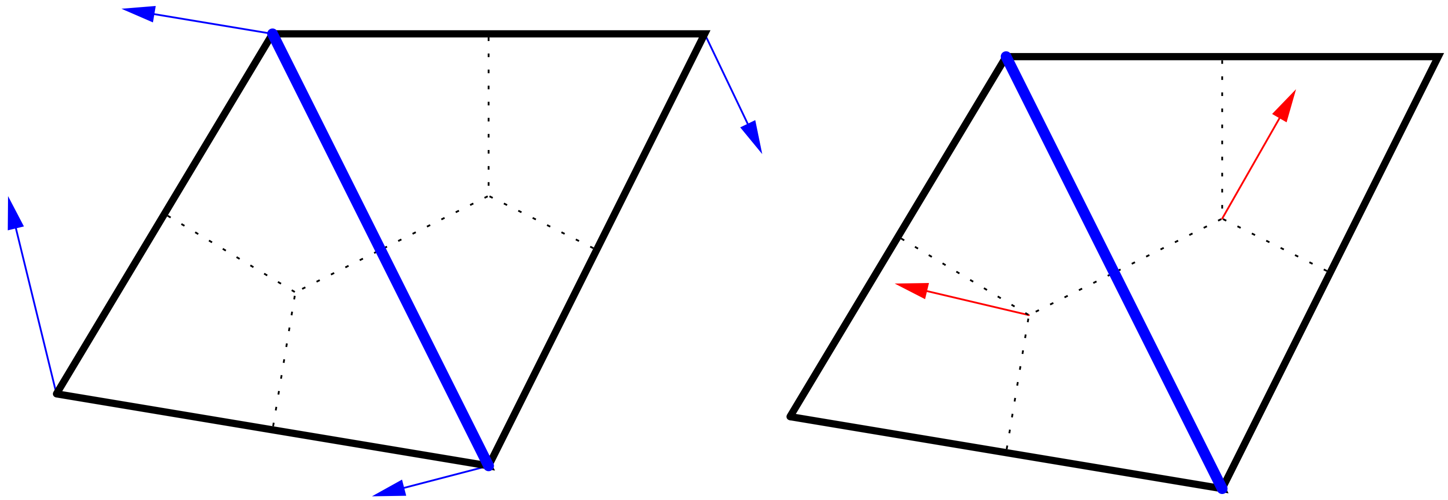
Primal-Primal Flats



Dual-Primal Flats



Proliferation of Discrete Flats



Types of data	Types of interpolation	Types of destination	Total
2	2	2	8

- Destination primal : $\flat_{\mathrm{ppp}}$, $\flat_{\mathrm{pdp}}$, $\flat_{\mathrm{dpp}}$, $\flat_{\mathrm{ddp}}$,
- Destination dual : $\flat_{\mathrm{ppd}}$, $\flat_{\mathrm{pdd}}$, $\flat_{\mathrm{dpd}}$, $\flat_{\mathrm{ddp}}$.

Derivation of Flat Operators

Example : DPP Flat

Let X be a smooth vector field on a smooth Riemannian manifold M and r a smooth curve on M of length L.

$$\int_r X^\flat = \int_{t_a}^{t_b} X(r(t)) \cdot \dot{r}(t)\, dt \qquad \text{(by definition)}$$

$$= \int_0^L X(r(s)) \cdot \hat{r}(s)\, ds \qquad \text{(parametrization indep.)}$$

Derivation of Flat Operators

Example : DPP Flat

Let X be a constant vector field on affine space M and r a straight line on M.

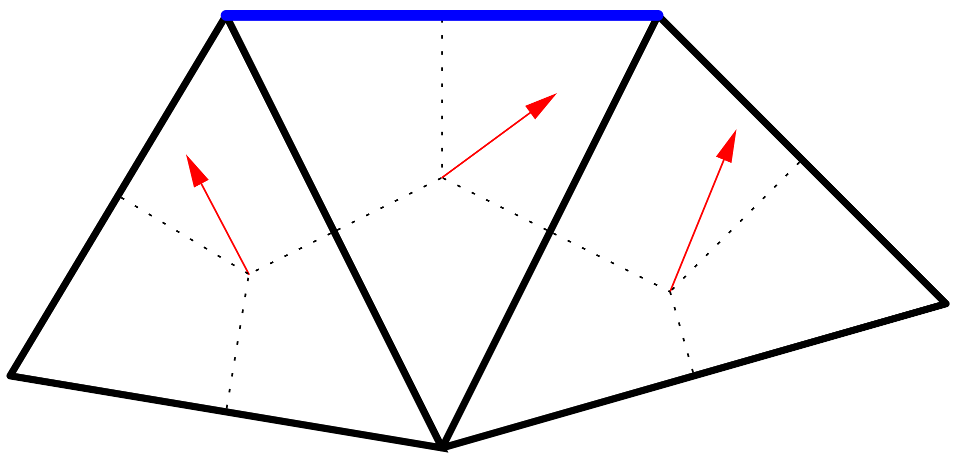
$$\int_r X^\flat = \int_0^L X \cdot \hat{r}(s)\, ds$$

$$= X \cdot \hat{r}(0)\, L = X \cdot \vec{r}$$

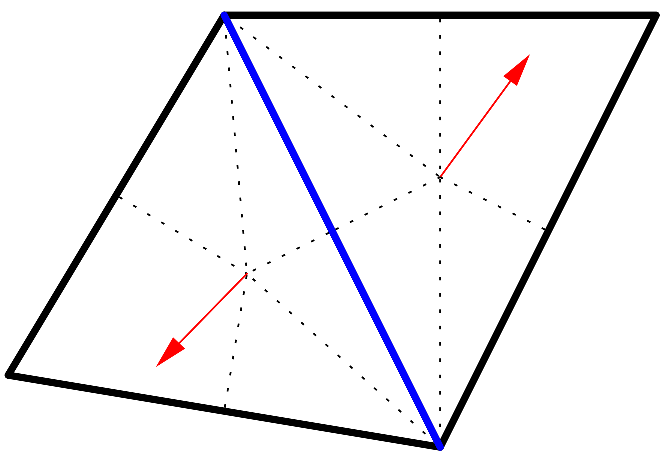
Derivation of Flat Operators

Example : DPP Flat (contd.)

$$\int_{r} X^{\flat} = X \cdot \hat{r}(0)\, L = X \cdot \vec{r}$$

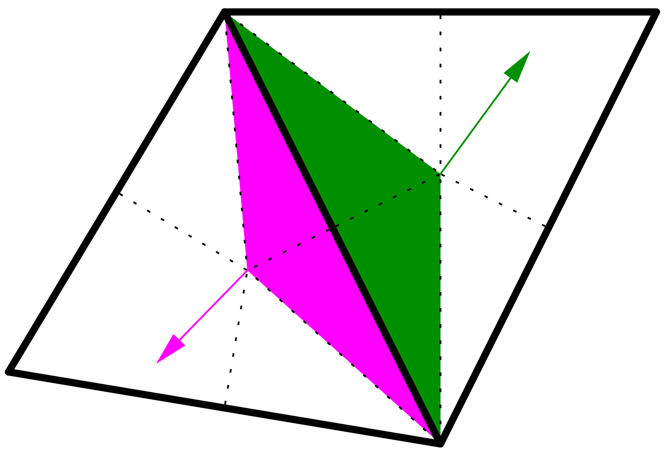


Easy case

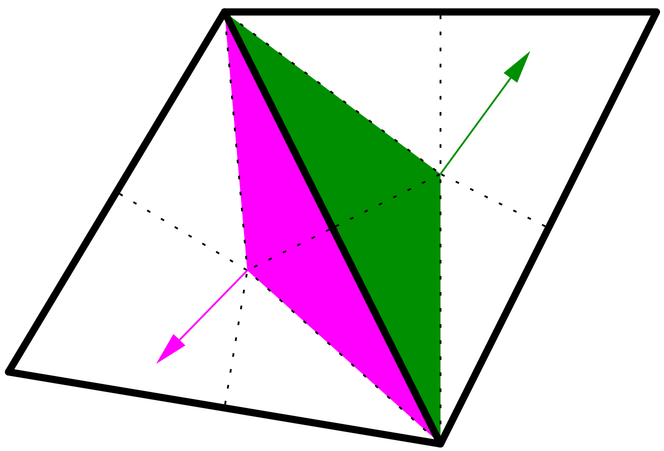


Interesting case

DPP Flat



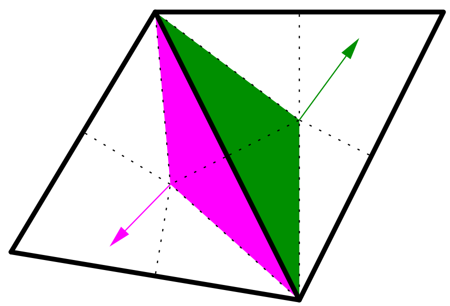
DPP Flat



- The colored region has a name : support volume.

DPP Flat

- Take a weighted sum of the two vectors.
- Weighting factor = area of one small colored triangle divided by total area of small colored triangles.
- Equivalent to fraction of dual edge in each triangle.

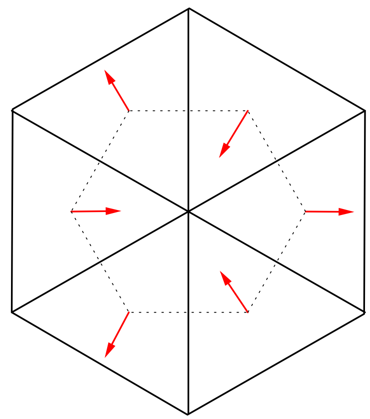


$$\langle X^\flat, \sigma^1 \rangle = \sum_{\sigma^n \succ \sigma^1} \frac{| \star \sigma^1 \cap \sigma^n |}{| \star \sigma^1 |} X(\star \sigma^n) \cdot \vec{\sigma}^1$$

DPP Flat

Lack of inverse

Proposition. *The discrete flat $\flat_{\mathrm{dpp}}$ is neither surjective nor injective. Thus it does not even have a one-sided inverse.*



Uniqueness

- The weighting factors are the unique factors such that discrete divergence theorem is satisfied.

A Primal-Dual Sharp for Exact Forms : PD Grad

- A simple calculation in a triangle using linear basis functions ϕ_0, ϕ_1 and ϕ_2.

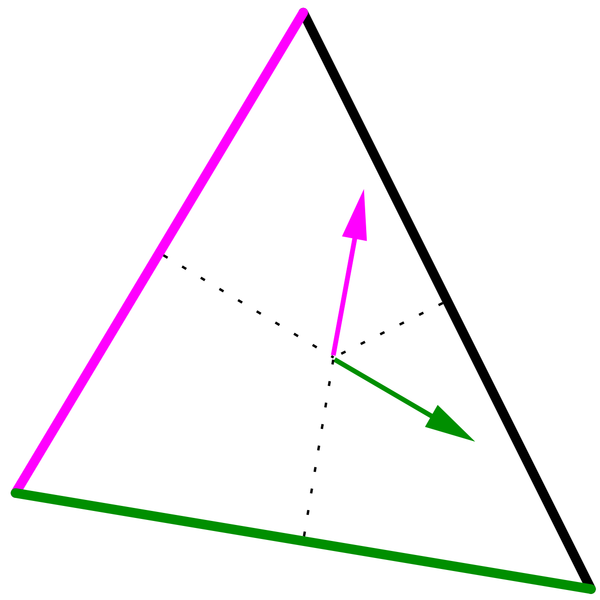
$$f = f_0\phi_0 + f_1\phi_1 + f_2\phi_2$$

$$\begin{aligned}\nabla f &= f_0\nabla\phi_0 + f_1\nabla\phi_1 + f_2\nabla\phi_2\\ &= -f_0\nabla\phi_1 - f_0\nabla\phi_2 + f_1\nabla\phi_1 + f_2\nabla\phi_2\\ &= (f_1 - f_0)\nabla\phi_1 + (f_2 - f_0)\nabla\phi_2\,.\end{aligned}$$

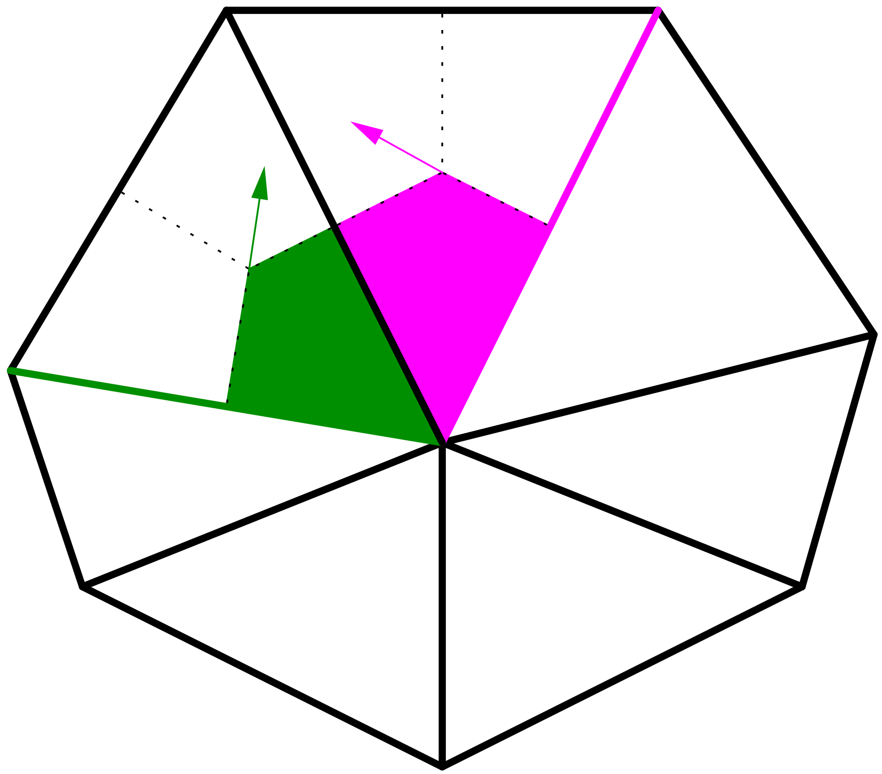
- Formula involves 1-form values $\mathbf{d}f$ and normals.
- Can be generalized to a primal-dual sharp for exact forms in any dimension.

A Primal-Dual Sharp for Exact Forms : PD Grad

$$\nabla f = (\mathbf{d}f)^{\sharp} = (f_1 - f_0)\nabla\phi_1 + (f_2 - f_0)\nabla\phi_2$$



A Primal-Primal Sharp



$$\alpha^\sharp(\sigma^0) = \sum_{[v,\sigma^0]} \langle \alpha, [v, \sigma^0] \rangle \sum_{\sigma^n \succeq [v,\sigma^0]} \frac{|\star\sigma^0 \cap \sigma^n|}{|\sigma^n|} \nabla \phi_{v,\sigma^n}$$

Separation of Forms and Vector Fields

- Common identifications made in this field :

$$F = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}$$

$$F^\flat = F_1 dx + F_2 dy + F_3 dz$$

$$*(F^\flat) = F_3 dx \wedge dy - F_2 dx \wedge dz + F_1 dy \wedge dz$$

- But that should not be done in a general theory since :

$$(\pounds_X \alpha)^\sharp \neq \pounds_X (\alpha^\sharp)$$

$$(*(\pounds_X \beta))^\sharp \neq \pounds_X ((*\beta)^\sharp)$$

Divergence

- Define discrete divergence by $\operatorname{div}(X) = -\delta X^\flat$.
- Discrete divergence theorem is satisfied :

$$|\star\sigma^0| \;\; \left\langle \operatorname{div}(X), \sigma^0 \right\rangle = \sum_{\sigma^1 \succ \sigma^0} \;\; \sum_{\sigma^n \succ \sigma^1} |\star\sigma^1 \cap \sigma^n| \;\; \left(X(\star\sigma^n) \cdot \frac{\vec{\sigma}^1}{|\sigma^1|} \right) .$$

- The 2D version of this appears in Polthier and Preuss [2002].
- Preliminary "thoughts" on the Lie derivative definition of divergence : $\operatorname{div}(X)\mu = \pounds_X \mu$.

Curl in 3D

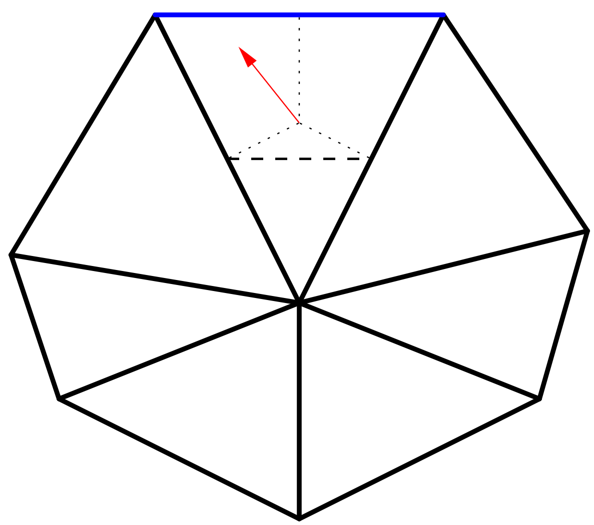
- Define 3D curl by $\text{curl}(X) = (*\mathbf{d}(X^\flat))^\sharp$.
- Dual-primal 3D curl satisfying usual vector calculus identities found in our other work (Tong et al. [2003 (to appear]). No derivation yet in DEC.

Curl in 2D

- Dual-primal 2D curl in DEC ($*dX^\flat$) matches one found by Polthier and Preuss [2002]:

$$\left\langle \mathrm{curl}(X), \sigma^0 \right\rangle = \frac{1}{2} \sum_{\sigma^2 \succ \sigma^0} X(\star\sigma^2) \cdot \vec{\sigma}^1(\sigma^2)$$

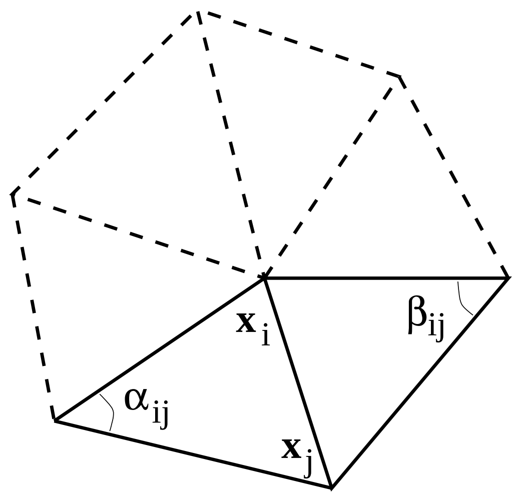
where $\sigma^1(\sigma^2)$ is the outer edge of triangle σ^2.



Laplace-Beltrami

- In computer graphics recently Meyer et al. [2002] found this formula for Laplace-Beltrami for a 2D triangle mesh embedded in 3D :

$$\Delta f(\mathbf{x}_i) = \frac{1}{2\mathcal{A}} \sum_{j \in N_1(i)} (\cot\ \alpha_{ij} + \cot\ \beta_{ij})\ \ (f(\mathbf{x}_i) - f(\mathbf{x}_j))$$



Laplace-Beltrami

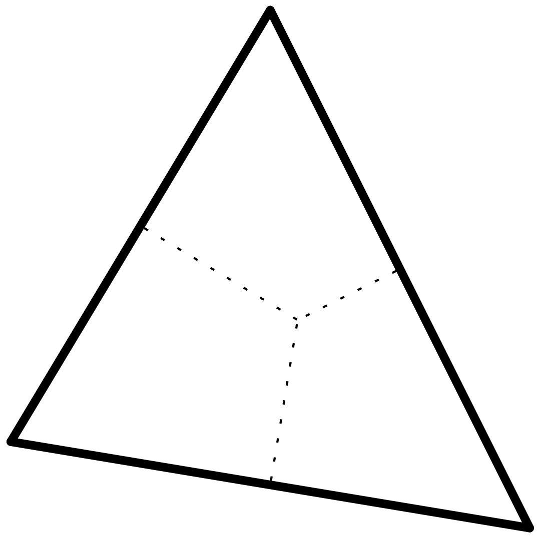
- By using definitions of DEC we find exactly the same formula. In DEC notation it is :

$$\langle \Delta f, \sigma^0 \rangle = -\frac{1}{|\star\, \sigma^0|} \sum_{\sigma^1 = [\sigma^0, v]} \frac{|\star\, \sigma^1|}{|\sigma^1|} (f(v) - f(\sigma^0))\,.$$

Wedge Product

Primal-Primal Wedge

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \langle \alpha^k, [v_{\tau(0)}, \ldots, v_{\tau(k)}] \rangle \langle \beta^l, [v_{\tau(k)}, \ldots, v_{\tau(k+l)}] \rangle$$



Wedge Product

Primal-Primal Wedge

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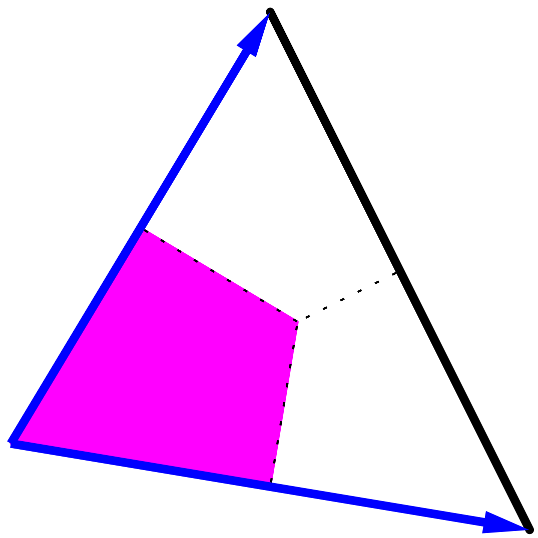

Wedge Product

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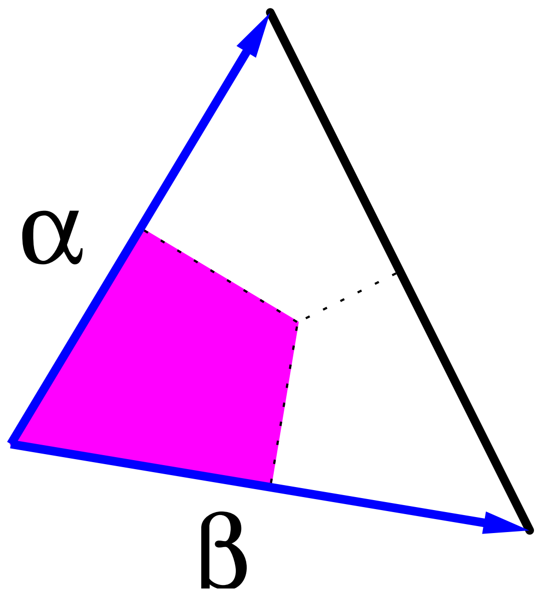
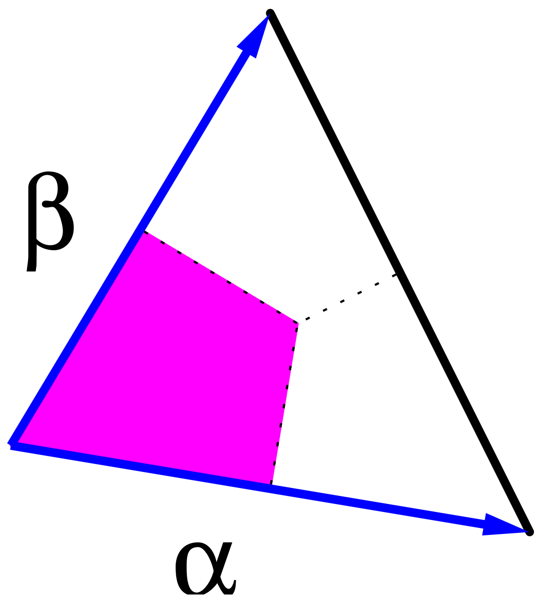


Wedge Product

Primal-Primal Wedge

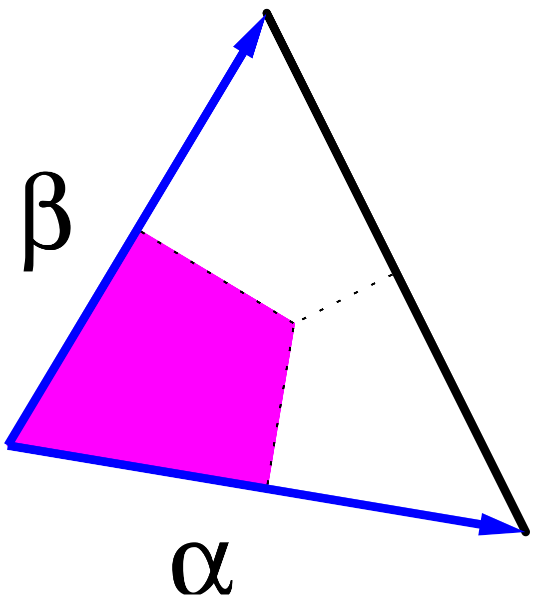
$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \mathrm{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \langle \alpha^k, [v_{\tau(0)}, \dots, v_{\tau(k)}] \rangle \langle \beta^l, [v_{\tau(k)}, \dots, v_{\tau(k+l)}] \rangle$$



Wedge Product

Primal-Primal Wedge

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \text{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \langle \alpha^k, [v_{\tau(0)}, \ldots, v_{\tau(k)}] \rangle \langle \beta^l, [v_{\tau(k)}, \ldots, v_{\tau(k+l)}] \rangle$$



- And so on for other corners.

Wedge Product

Properties

(i) **Anti-commutativity** $\alpha^k \wedge \beta^k = (-1)^{kl} \beta^l \wedge \alpha^k$.

(ii) **Leibniz rule** $\mathbf{d}(\alpha^k \wedge \beta^l) = (\mathbf{d}\alpha^k) \wedge \beta^l + (-1)^k \alpha^k \wedge (\mathbf{d}\beta^l)$.

(iii) **Associativity for closed forms** For $\alpha^k \in C^k(K)$, $\beta^l \in C^l(K)$, $\gamma^m \in C^m(K)$, such that $\mathbf{d}\alpha^k = 0$, $\mathbf{d}\beta^l = 0$, $\mathbf{d}\gamma^m = 0$, we have that, $(\alpha^k \wedge \beta^l) \wedge \gamma^m = \alpha^k \wedge (\beta^l \wedge \gamma^m)$.

Interior Product (a.k.a Contraction)

Algebraic definition

- We proved the following identity of smooth exterior calculus :

$$i_X \alpha = (-1)^{k(n-k)} * (*\alpha \wedge X^\flat) \, .$$

- This can be used to *define* i_X.

Interior Product

Dynamic definition

- Following identity is true in smooth case – given a smooth manifold M, vector field X on M and submanifold S :

$$\int_S i_X \beta = \left.\frac{d}{dt}\right|_{t=0} \int_{E_X^S(t)} \beta$$

- Here $E_X^S(t)$ is region swept out by flowing a submanifold S along the flow of X for time t.
- This leads to a combinatorial definition.

Lie Derivative

- Dynamic will be based on the following smooth case result we proved. The combinatorial formula not worked out yet.

$$\int_S \pounds_X \beta = \left.\frac{d}{dt}\right|_{t=0} \int_{S_t} \beta$$

- Algebraic via Cartan homotopy formula

$$\pounds_X \omega = i_X \mathbf{d}\omega + \mathbf{d} i_X \omega$$

- Possible problem with algebraic definition : the Leibniz rule might be only satisfied for closed forms (recall wedge properties).

Other Work

Template Matching

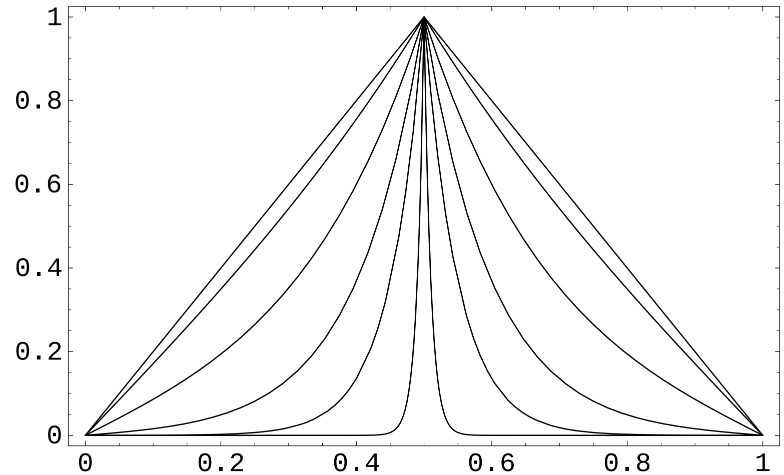
- PDEs for matching images inspired by averaging in fluid mechanics.
- Find shortest curve on diffeomorphism group.
- Different metrics on diffeomorphism group give different PDEs.
- Averaged Template Matching Equation in spatial dimension n

$$\mathbf{v}_t + (\operatorname{div}\mathbf{u})\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{Du})^T\mathbf{v} = 0$$
$$\mathbf{v} = (1 - \alpha^2\Delta)\mathbf{u}$$

- See Hirani et al. [2001].

Other Work

- **Template Matching** Joint work with S. Jon Chapman.
  - Initial conditions derived from natural boundary conditions :



  - The equations can be written in div, grad, curl form (Holm [2003]) and mimetic differencing has been applied to compute soliton strings solutions by Darryl Holm and Martin Staley at Los Alamos.

Other Work

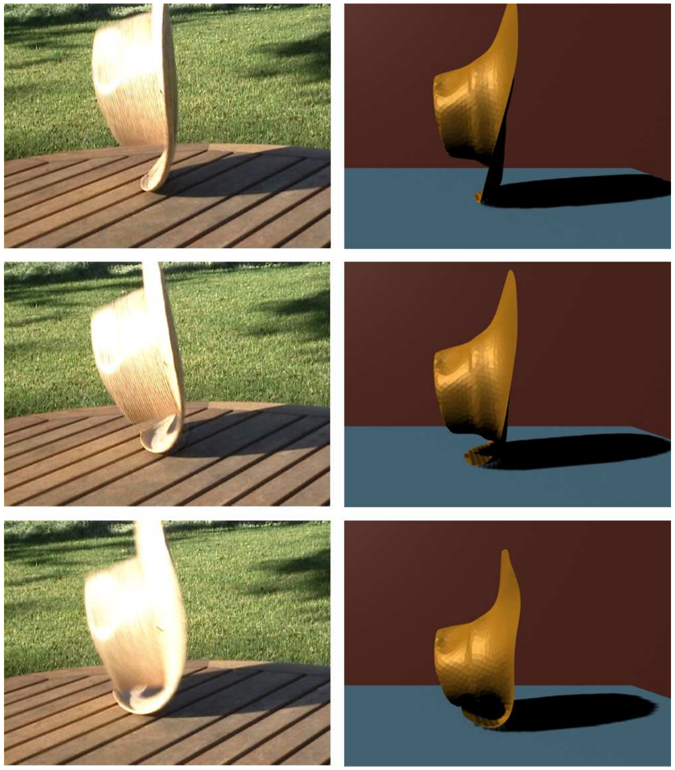
Discrete Shells

Joint work with E. Grinspun, M. Desbrun, P. Schröder.

- Shells : one dimension much smaller.
- Given triangle mesh *is* the surface.
- Define stored energy function discretely.

Other Work

■ Discrete Shells



Joint with E. Grinspun, M. Desbrun, P. Schröder. Computations by E. Grinspun. To appear in ACM SIGGRAPH/Eurographics Symposium on Computer Animation (SCA), July 2003.

Other Work

■ Multiscale Vector Field Processing in 3D

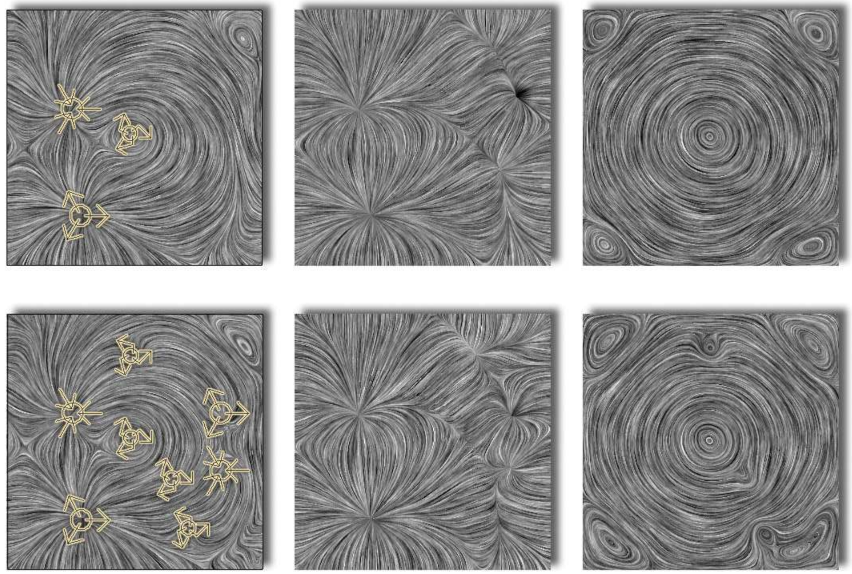
Joint work with Y. Tong, S. Lombeyda, M. Desbrun.

- Discrete 3D dual vector field given in tetrahedra.
- We give Hodge decomposition of given field.
- Applications in incompressible fluid mechanics and in incompressible elasticity.
- We also find the potentials and so processing possible.
- Discrete div, grad, curl satisfy usual identities :

$$\begin{aligned} \operatorname{div} \circ \operatorname{curl} &= 0 \\ \operatorname{curl} \circ \operatorname{grad} &= 0 \\ \operatorname{div} \circ \operatorname{grad} &= \Delta \end{aligned}$$

Other Work

Multiscale Vector Field Processing in 3D

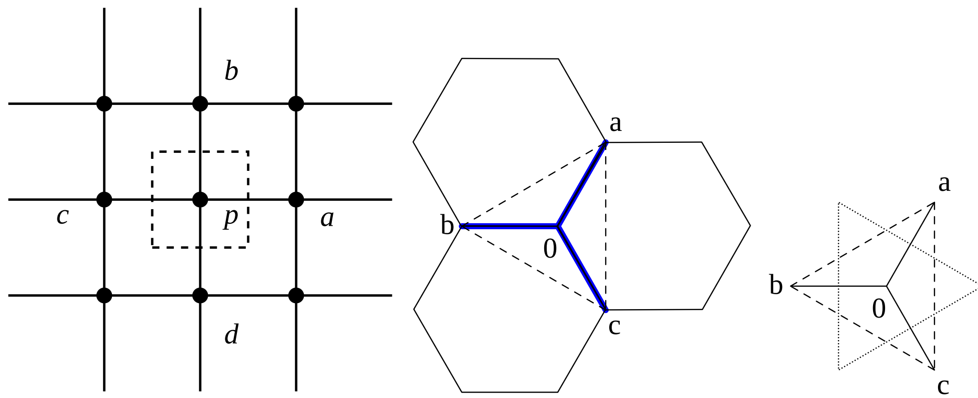


Joint work with Y. Tong, S. Lombeyda, M. Desbrun. Computations by Y. Tong and S. Lombeyda. 2D example shown for clarity. To appear in ACM Transactions on Graphics (SIGGRAPH 2003).

Speculative Work

Lattices

Square and Hexagonal Grid



Getting General Tensors Into DEC ?

A Primal-Primal Natural Pairing

$$\alpha(X)(\sigma^0) = \alpha^\sharp(\sigma^0) \cdot (X(\sigma^0))$$

Use Tensor Product of 1-forms to Define Tensors ?

$$(\alpha^1 \otimes \beta^1)(U, V) = \frac{1}{2}[\alpha(U)\beta(V) + \alpha(V)\beta(U)]$$

Conclusions

Conclusions

- We showed that by choosing approproate geometric and combinatorial defintions we can start building a discrete theory of extrior calculus.
- Discrete theory parallels smooth theory but has a variety of operators due to duality.
- There are enough promising hints to make implementation worthwhile. Many existing important formulas are reproduced by simple algebraic operations of DEC.

Future Work : Immediate

Fill Important Gaps in Current DEC Theory

- What is the DEC derivation of the curl we found in vector field work ?
- Which of the various sharps and flats combine to have nice properties and which don't ?
- Shouldn't a vector be a pair of points as in discrete mechanics ?
- Can we remove the restriction of flatness of mesh for primal vector fields ?
- Continue work on discrete pullback.

Future Work : Next on List

Implementation, convergence, applications

- This summer I'll be working on implementing DEC and its applications with 2 SURF students.
- We are trying to formulate convergence questions in DEC with the help of some researchers.
- Extensions for and applications to lattice theory are being considered by some collaborators.
- Applications to electromagnetism started by some collaborators.

Future Work : Research Program

- We want to extend the DEC theory :
  - General discrete tensors,
  - Multiscale meshes,
  - General cell complexes.
- Links with discrete mechanics : relationship between multisymplectic geometry and DEC.

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References

Douglas N. Arnold. Differential complexes and numerical stability. 2003.

Alain Bossavit. Generalized finite differences in computational electromagnetics. In F. L. Teixeira, editor, *Geometric Methods for Computational Electromagnetics*, chapter 2. EMW Publishing, 2001. URL `http://ceta-mac1.mit.edu/pier/pier32/pier32.html`.

Alain Bossavit. On generalized finite difference : a discretization of electromagnetic problems. 2002.

Aleksei A. Dezin. *Multidimensional analysis and discrete models*. CRC Press, Boca Raton, FL, 1995. ISBN 0-8493-9425-2. Translated from the Russian by Irene Aleksanova.

P.W. Gross and P.R. Kotiuga. Data structures for geometric and topological aspects of finite element algorithm. In F. L. Teixeira, editor, *Geometric Methods for Computational Electromagnetics*, chapter 6. EMW Publishing, 2001. URL `http://ceta-mac1.mit.edu/pier/pier32/pier32.html`.

Xianfeng Gu. *Parametrization for surfaces with arbitrary topology*. PhD thesis, Harvard University, 2002.

Jenny Harrison. Stokes' theorem for nonsmooth chains. *Bull. Amer. Math. Soc. (N.S.)*, 29(2):235–242, 1993. ISSN 0273-0979.

R. Hiptmair. Discrete Hodge operators. *Numer. Math.*, 90(2):265–289, 2001. ISSN 0029-599X.

R. Hiptmair. Discretization of maxwell's equations. In *Numerical Relativity Workshop*, 2002.

Anil N. Hirani, Jerrold E. Marsden, and James Arvo. Averaged template matching equations. In *Energy minimization methods in computer vision and pattern recognition, LNCS 2134*, pages 528–543. Springer-Verlag, 2001. URL `http://www.cs.caltech.edu/ hirani/research`.

Darryl D. Holm. Personal communication. 2003.

J. M. Hyman and M. Shashkov. Natural discretizations for the divergence, gradient, and curl on logically rectangular grids. *Comput. Math. Appl.*, 33(4):81–104, 1997a. ISSN 0898-1221.

James M. Hyman and Mikhail Shashkov. Adjoint operators for the natural discretizations of the divergence, gradient and curl on logically rectangular grids. *Appl. Numer. Math.*, 25(4):413–442, 1997b. ISSN 0168-9274.

Elizabeth L. Mansfield and Peter E. Hydon. On a variational complex for difference equations. In *The geometrical study of differential equations (Washington, DC, 2000)*, volume 285 of *Contemp. Math.*, pages 121–129. Amer. Math. Soc., Providence, RI, 2001.

Claudio Mattiussi. An analysis of finite volume, finite element, and finite difference methods using some concepts from algebraic topology. *Journal of Computational Physics*, 1997.

Mark Meyer, Mathieu Desbrun, Peter Schröder, and Alan H. Barr. Discrete differential-geometry operators for triangulated 2-manifolds. In *International Workshop on Visualization and Mathematics, VisMath*, 2002.

Konrad Polthier and E. Preuss. Identifying vector field singularities using a discrete hodge decomposition. In H. C. Hege and Konrad Polthier, editors, *Visualization and Mathematics, VisMath*. Springer-Verlag, 2002.

W. Schwalm, B. Moritz, M. Giona, and M. Schwalm. Vector difference calculus for physical lattice models. *Phys. Rev. E (3)*, 59(1, part B):1217–1233, 1999. ISSN 1063-651X.

Samik Sen, Siddhartha Sen, James C. Sexton, and David H. Adams. Geometric discretization scheme applied to the abelian Chern-Simons theory. *Phys. Rev. E (3)*, 61(3):3174–3185, 2000. ISSN 1063-651X.

F. L. Teixeira. Geometric aspects of the simplicial discretization of maxwell's equations. In F. L. Teixeira, editor, *Geometric Methods for Computational Electromagnetics*, chapter 7. EMW Publishing, 2001. URL `http://ceta-mac1.mit.edu/pier/pier32/pier32.html`.

Yiying Tong, Santiago Lombeyda, Anil N. Hirani, and Mathieu Desbrun. Discrete multiscale vector field decomposition. *ACM Transactions on Graphics (SIGGRAPH 2003 issue)*, 2003 (to appear).

E. Tonti. Finite formulation of electromagnetic field. *IEEE Trans. Mag.*, 38:333–336, 2002.

Joe Warren, Scott Schaefer, Anil N. Hirani, and Mathieu Desbrun. Barycentric coordinates for convex sets. 2003.

Backup Slides

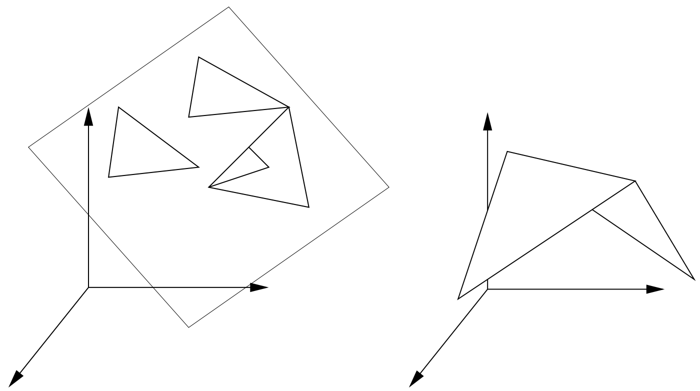
Primal and Dual Complexes

Orienting the Primal Complex

- **Orientation** is class of vertex permutation
- Equivalent to **corner** or **polyline** basis
- **Induced** orientation on $(n-1)$-simplex
- Affine space called **plane** of a simplex $P(\sigma^p)$
- Manifold-like simplcial complex
- Comparing orientations

Primal and Dual Complexes

■ Orienting the Primal Complex



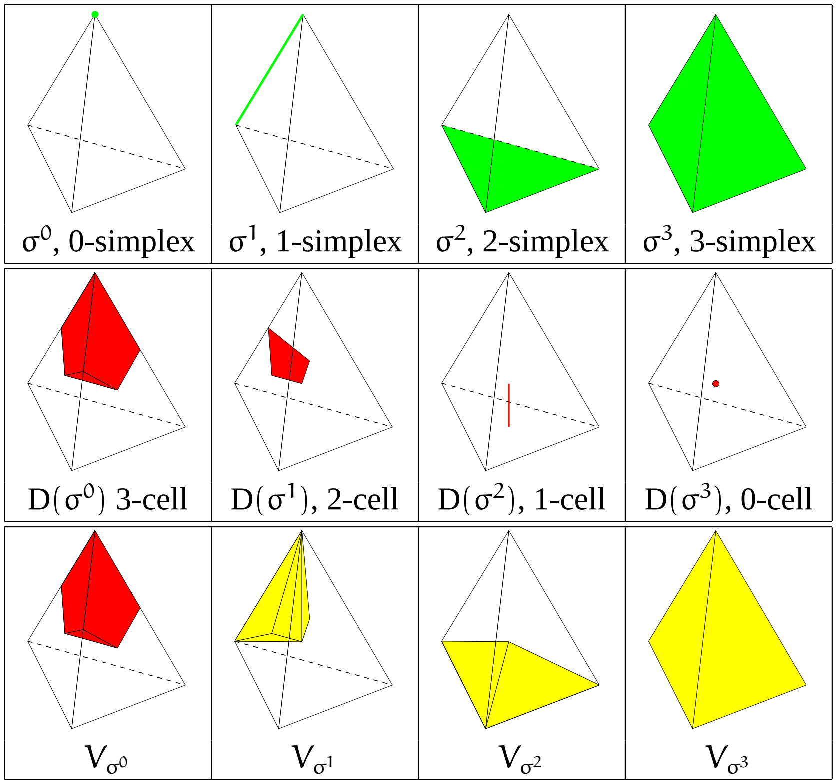
Primal and Dual Complexes

Forming the Dual Complex

- First **subdivide** primal, then aggregate smaller simplices into **dual cells**
- We use **circumcentric** subdivision
- Restriction : **fat** manifold-like simplicial complexes
- Dual cells written as $D(\sigma^0)$, $D(\sigma^1)$, $D(\sigma^2)$ etc.

Primal and Dual Complexes

Forming the Dual Complex



Primal and Dual Complexes

Orienting the Dual Complex

- **Subdivision simplex** and **elementary dual cell**
- Transversailty of the two
- Identical planes of various pieces

Primal and Dual Complexes

Orienting the Dual Complex

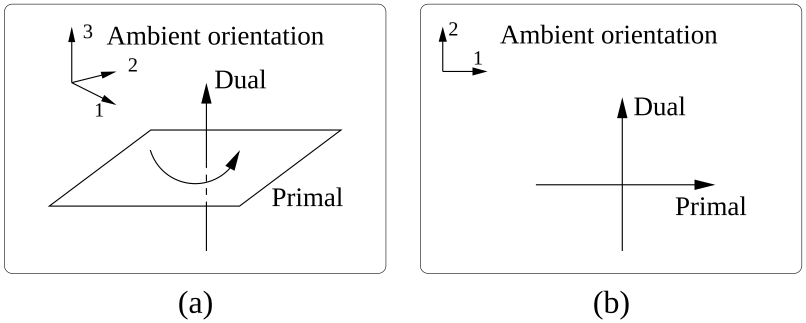


FIGURE 1: Relationship between orientations of embedding space, embedded “primal” manifold and an embedded “dual” manifold transverse to the primal

Primal and Dual Complexes

Orienting the Dual Complex

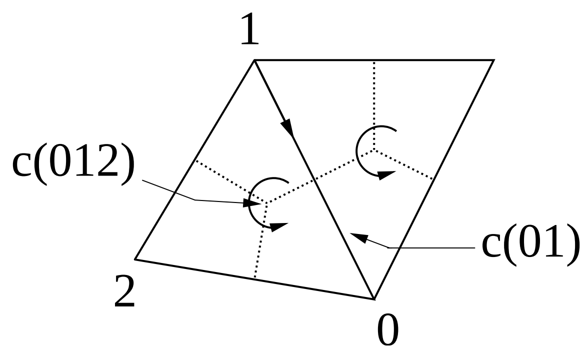


FIGURE 2: Orienting an elementary dual simplex.

Primal and Dual Complexes

Orienting the Dual Complex

$$\text{primal simplex}: \sigma^p = \sigma^1 = [v_1, v_0]$$
$$\sigma^0 \prec \sigma^1 \prec \sigma^2 \text{ instance}: v_0 \prec [v_1, v_0] \prec [v_0, v_1, v_2]$$
$$\left[c\left(\sigma^0\right), c\left(\sigma^1\right)\right] = [v_0, c_{01}]$$
$$\text{elementary dual}: s\left[c\left(\sigma^1\right), c\left(\sigma^2\right)\right] = s[c_{01}, c_{012}]$$
$$\text{subdivision simplex}: \left[c\left(\sigma^0\right), c\left(\sigma^1\right)\right] = [v_0, c_{01}] .$$

Forms and Vector Fields

Discrete Vector Fields

A **discrete dual vector field** X on a fat manifold-like simplicial complex K is $X : K_{(0)} \to \mathbb{R}^N$ such that $X(D(\sigma^n)) \in P(\sigma^n)$. This space called $\mathfrak{X}_d(\star K)$.

Let K be a *flat* fat manifold-like simplicial complex of dimension n. A **discrete primal vector field** X is $X : K^{(0)} \to \mathbb{R}^n$. This space called by $\mathfrak{X}_d(K)$.